

POISSON-LIE T-DUALITY FOR QUASITRIANGULAR LIE BIALGEBRAS

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Abstract We introduce a new 2-parameter family of sigma models exhibiting Poisson-Lie T-duality on a quasitriangular Poisson-Lie group G . The models contain previously known models as well as a new 1-parameter line of models having the novel feature that the Lagrangian takes the simple form $\mathcal{L} = E(u^{-1}u_+, u^{-1}u_-)$ where the generalised metric E is constant (*not* dependent on the field u as in previous models). We characterise these models in terms of a global conserved G -invariance. The models on $G = SU_2$ and its dual G^* are computed explicitly. The general theory of Poisson-Lie T-duality is also extended; we develop the Hamiltonian formulation and the reduction for constant loops to integrable motion on the group manifold. Finally, we generalise T-duality in the Hamiltonian formulation to group factorisations $D = G \bowtie M$ where the subgroups need not be dual or even have the same dimension and need not be connected to the Drinfeld double or to Poisson structures.

1 Introduction

Poisson-Lie T-duality has been introduced in [1][2] and other works as a non-Abelian version of T-duality in string theory, based on duality of Lie bialgebras. A motivation (stated in [1]) is quantum group or Hopf algebra duality; this had been introduced as a duality for physics several years previously[3][4][5][6], as an ‘observable-state’ duality for certain quantum systems based on group factorisations $D = G \bowtie M$. In one system a particle moves in G under the action of M and its quantum algebra of observables is the bicrossproduct Hopf algebra $U(\mathfrak{m}) \bowtie \mathbb{C}(G)$, in the dual system the roles of G, M are interchanged but its quantum algebra of observables $\mathbb{C}(M) \bowtie U(\mathfrak{g})$

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has the same physical content but with the roles of observables/states and position/momentum interchanged (here $\mathfrak{g}, \mathfrak{m}$ are the Lie algebras of G, M respectively). Indeed, being mutually dual Hopf algebras the two quantum systems are related to each other by quantum Fourier transform

$$\mathcal{F} : U(\mathfrak{m}) \bowtie \mathbb{C}(G) \rightarrow \mathbb{C}(M) \bowtie U(\mathfrak{g}), \quad (1)$$

see [7] where this was recently studied in detail for the simplest example (the so-called Planck-scale Hopf algebra $\mathbb{C}[p] \bowtie \mathbb{C}[x]$ in [3].) Under this observable-state duality it was shown in [3] that one had inversion of coupling constants as well as connections with Planck-scale physics. At about the same time, Abelian T-duality was introduced in [8] and elsewhere as a momentum-winding mode symmetry in string theory with some similar features. The observable-state duality (1) is not, however, limited in any way to the Abelian case and indeed there is a natural model for every compact simple group G with $M = G^*$ the Yang-Baxter dual. Here a Lie bialgebra is an infinitesimal version of a Hopf algebra and has a dual \mathfrak{g}^* , and G^* is its associated Lie group. It is also the group of dressing transformations[9] in the theory of classical inverse scattering and the solvable group in the Isawasa decomposition $D = G_{\mathbb{C}} = G \bowtie G^*$ of the complexification of the compact Lie group G , see [6]. Moreover, $D = G \bowtie G^*$ is the Lie group associated to the Drinfeld double $\mathfrak{d}(\mathfrak{g})$ of \mathfrak{g} as a Lie bialgebra [10]. The Lie bialgebra structure of \mathfrak{g} also implies a natural Poisson bracket on G [10]. Further details are in the Preliminaries; see also [11] for an introduction to these topics. These quantum systems $U(\mathfrak{g}) \bowtie \mathbb{C}(G^*)$ with observable-state duality were constructed in [4][5][6] as one of the two main sources of quantum groups canonically associated to a simple Lie algebra (the other is the more well-known q -deformation of $U(\mathfrak{g})$ to quantum groups $U_q(\mathfrak{g})$).

The subsequent theory of Poisson-Lie T-duality[2] indeed has many of the same features. One system consists of a sigma model on the group G with a Lagrangian of the form

$$\mathcal{L} = E_u(u^{-1}u_+, u^{-1}u_-), \quad u : \mathbb{R}^{1,1} \rightarrow G,$$

where u is the field, u_{\pm} are derivatives in light-cone coordinates and E_u a bilinear form on \mathfrak{g} but depending on the value of u (a ‘generalised metric’ since E_u need not be symmetric). The dual theory is a sigma-model on G^* with

$$\hat{\mathcal{L}} = \hat{E}_t(t^{-1}t_+, t^{-1}t_-), \quad t : \mathbb{R}^{1,1} \rightarrow G^*.$$

The physical content of the two theories is established to be the same due to the existence of the larger group $D = G \bowtie G^*$ associated to the Drinfeld double $\mathfrak{d}(\mathfrak{g})$.

In the present paper we extend Poisson-Lie T-duality in several directions, motivated in part by the above connections with quantum groups and observable-state duality. From a physical point of view the main result is as follows: the previously-known models exhibiting Poisson-Lie T-duality require a very special form of the generalised metric E_u depending on u in a rather complicated way (related to the Poisson bracket on G). This is in sharp contrast to the usual principal sigma model[12] where the metric is a constant, the Killing form K . As a result, Poisson-Lie T-duality would appear to be somewhat artificial and to apply to only certain highly non-linear models where the ‘metric’ in the target group is far from constant. Even the explicit form of E_u is known only in some simple cases such as $\mathfrak{g} = \mathfrak{b}_+$ the Borel-subalgebra of \mathfrak{su}_2 [1]. The $\mathfrak{g} = \mathfrak{su}_2$ case was discussed recently in [13] but still without fully explicit formulae for the resulting Lagrangians. Our main result is the introduction of a new 2-parameter class of models within the existing general framework for Poisson-Lie T-duality but which much nicer properties. We also provide new computational tools using the theory of Lie bialgebras to compute the models explicitly. We obtain, for example, the explicit Lagrangians in the SU_2 case and its dual.

These new models require that \mathfrak{g} is a quasitriangular Lie bialgebra, i.e. defined by an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ obeying the so-called modified *classical Yang-Baxter equations*[10]. This includes all complex semisimple Lie algebras equipped, for example, with their standard Drinfeld-Sklyanin quasitriangular structure as used in the theory of classical inverse scattering. The quantisations of the associated Poisson bracket on G in these cases include coordinate algebras of the quantum groups $U_q(\mathfrak{g})$. This is therefore an important class of models, and we will find quite tractable formulae in this case. We use r not only in the Lie bialgebra structure (which is usual) but again in certain boundary conditions for the graph coordinates in order to cancel their natural u -dependence for the choice of certain parameters. This greater generality allows for a two-parameter family of models associated to this data. Moreover, in this extended parameter space there is a novel line of ‘nice’ models in which $E_u = E_e$ is a *constant* not dependent at all on u . This line includes at ∞ the standard principal sigma model where $E_e = K$ the Killing form, but at other points has an antisymmetric part built from r itself. In this way one may approach the

principal sigma model itself along a line of sigma models exhibiting Poisson-Lie T-duality and of a simple form without additional non-linearities due to a non-constant generalised metric. The dual models are more complicated but at ∞ , for example, one obtains an Abelian model as the Poisson-Lie T-dual of the principal sigma model approached in this way (the latter lies on the boundary of the space of models exhibiting T-duality). These results are presented in Section 6.

Also in the paper we develop the Hamiltonian picture of Poisson-Lie T-duality in rather more detail than we have found elsewhere; see also [14]. This is done in Section 3 after the preliminary Section 2. Among the new results is a more regular expression for the Hamiltonian that covers both the model and the dual model simultaneously. Also new is a study of the symmetries of the theory induced by the *left* action of D on itself. These are not usually considered because they are not conserved but we show that they do respect the symplectic structure. Moreover, when E_u is constant we show that the action of $G \subset D$ is conserved and we compute the conserved charges.

A second general development, in Section 4, is a study of the classical mechanical system on G (say) in the limit of point-like strings (i.e. x -independent solutions). We show that this constraint commutes with the dynamics and we provide the resulting Lagrangian and Hamiltonian systems and the phase space. The left action of D descends to the classical mechanical system and we show that it has a moment map. The conserved charges are computed in the case of constant E_u . The dual model on G^* equivalent to these point-solutions are not point solutions but extended solutions of a certain special form. We also discuss the quantisation of this classical mechanical system both conventionally and in a manner relevant to the conserved charges. Although these systems appear to be different from the systems $U(\mathfrak{g}^*) \bowtie \mathbb{C}[G]$ exhibiting observable-state duality at the Planck-scale[3], we do establish some points of comparison, such as a common phase space.

Section 5 contains some further algebraic preliminaries needed for the explicit construction of E_u . We show that

$$\text{Ad}_u^*(E_u) = (E_e^{-1} + \Pi(u))^{-1}$$

where Π is the $\mathfrak{g} \otimes \mathfrak{g}$ -valued function defining the Poisson-structure on G . To our knowledge this derivation differs from previous work in that we do not assume anything about E_e^{-1} , in particular it need not be the Killing form usually added[2] to Π as an ansatz. This greater generality allows us in Section 6 to present our main result; the class of ‘nice’ Poisson-Lie T-dual models based

on quasitriangular Lie bialgebras.

Finally, Section 7 introduces new ‘double-Neumann’ boundary conditions for the open string and proceeds for these (as well as more trivially for closed strings) to extend the Poisson-Lie T-duality in the Hamiltonian form to general group factorisations $D = G \bowtie M$, where D need no longer be the Lie group of the Drinfeld double $\mathfrak{d}(\mathfrak{g})$ and indeed \mathfrak{m} need not be \mathfrak{g}^* but could be some quite different Lie algebra, possibly of different dimension. This is directly motivated by the observable-state duality models which exist[4][11] for any factorisation. It is also motivated by the Adler-Kostant-Symes theorem in classical inverse scattering which works for a general factorisation equipped with an inner product, see [11]. The dynamics are determined, similarly to the conventional bialgebra theory, by the splitting of the Lie algebra of D into orthogonal subspaces but these need no longer be of the same dimension (although only in this case is there a sigma-model interpretation). We also have an action of D by left multiplication on the phase space with the double-Neumann boundary conditions which is useful even for standard Poisson-Lie T-duality based on Lie bialgebras. In particular, it extends to an action of the affine Kac-Moody Lie algebra $\tilde{\mathfrak{d}}$.

Several directions remain for further work. First of all, only some first steps are taken (in Section 4) to relate T-duality to observable-state duality (1) in the quantum theory; our long term motivation here is to extend these ideas from particles to loops and hence to formulate T-duality for the full quantum systems as a duality operation on a more general algebraic structure (no doubt more general than Hopf algebras but in the same spirit). This in turn would give insight into the correct algebraic structure for the conjectured ‘M-theory’ about which little is known beyond dualities visible in the Lagrangians at various classical limits. Let us mention only that Poisson-Lie T-duality is connected also with mirror symmetry[15] and indirectly with several other relevant dualities in the theory of strings and branes.

Secondly, there are some interesting examples of the generalisation of Poisson-Lie T-duality in Section 7 which exist in principle and should be developed further. Thus, the conformal group on \mathbb{R}^n ($n > 2$) has, locally, a factorisation into the Poincaré group and an \mathbb{R}^n of special conformal translations. The global structure of the factorisation is singular in a similar manner to the ‘black-hole event-horizon’-like features of the Planck-scale Hopf algebra $\mathbb{C}[p] \bowtie \mathbb{C}[x]$ in [3]. There is also the possibility in our more general setting of a many-sided T-duality (i.e. not only

two equivalent theories) associated to more than one factorisation of the same group.

Finally, the natural emergence of generalised metrics which have both symmetric and anti-symmetric parts is a natural feature of noncommutative Riemannian geometry[16] (where symmetry is natural only in the commutative limit). This is a further direction that remains to be explored. Also to be considered is the addition of WZNW terms to render our 2-parameter class of sigma-models conformally invariant as well as the computation of 1-loop or higher quantum effects c.f. [17] [18].

Preliminaries

We recall, see e.g.[11] that a Lie bialgebra is a Lie algebra equipped with $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ where δ is antisymmetric and obeys the coJacobi identity (so that \mathfrak{g}^* is a Lie algebra) and

$$\delta[\xi, \eta] = \text{ad}_\xi(\eta) - \text{ad}_\eta(\xi)$$

for all $\xi, \eta \in \mathfrak{g}$, where ad extends as a derivation.

Next, associated to any Lie bialgebra \mathfrak{g} there is a double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^{*\text{op}}$. This is a double semidirect sum with cross relations

$$[\phi, \xi] = \phi \triangleright \xi - \phi \triangleleft \xi$$

where the actions are mutually coadjoint ones

$$\phi \triangleright \xi = \langle \xi_{[2]}, \phi \rangle \xi_{[1]}, \quad \phi \triangleleft \xi = \langle \xi, \phi_{[2]} \rangle \phi_{[1]}$$

where the angle brackets are the dual pairing of \mathfrak{g}^* with \mathfrak{g} and $\delta(\xi) = \xi_{[1]} \otimes \xi_{[2]}$. Here \mathfrak{d} is quasitriangular and factorisable (see later) and as a result there is an adjoint invariant inner product on \mathfrak{d} ,

$$(\xi \oplus \phi, \eta \oplus \psi) = \langle \phi, \eta \rangle + \langle \phi, \xi \rangle .$$

Here

$$\mathfrak{g}^* = \mathfrak{g}^{*\text{op}} \tag{2}$$

and \mathfrak{g} are maximal isotropic subspaces. We will need this description from [5] which is somewhat more explicit than the usual description in terms of ‘Manin triple’ in Drinfeld’s work [19].

Given a double cross sum of Lie algebras $\mathfrak{g} \bowtie \mathfrak{m}$, we may at least locally exponentiate to a double cross product of Lie groups $G \bowtie M$. This is given explicitly in [6]. We view the Lie

algebra actions as cocycles, exponentiate to Lie group cocycles, view these as flat connections and take the parallel transport operation. The actions can be described by $b(u) \in \mathfrak{g} \otimes \mathfrak{m}^*$ given by $b(u)(\phi) = b_\phi(u) = (\phi \triangleright u)u^{-1}$ and $a(s) \in \mathfrak{g}^* \otimes \mathfrak{m}$ given by $a(s)(\xi) = a_\xi(s) = s^{-1}(s \triangleleft \xi)$. It can be shown that $b \in Z_{\text{Ad} \otimes \triangleleft^*}^1(G, \mathfrak{g} \otimes \mathfrak{m}^*)$ is a cocycle, where the action \triangleleft^* is a left action of G on \mathfrak{m}^* given by dualising the right action $\triangleleft : \mathfrak{m} \times G \rightarrow \mathfrak{m}$. Also $a \in Z_{\triangleright^* \otimes \text{Ad}_R}^1(M, \mathfrak{g}^* \otimes \mathfrak{m})$, where Ad_R is the right adjoint action of M on \mathfrak{m} and \triangleright^* is the right action of M on \mathfrak{g}^* given by dualising its action on \mathfrak{g} . These Lie-algebra-valued functions a, b generate the vector fields for the action of \mathfrak{g} on M and \mathfrak{m} on G respectively. Thus, $\phi \triangleright u = b_\phi(u)u$ where $\xi u = \tilde{\xi}$ denotes the right invariant vector field on G generated by $\xi \in \mathfrak{g}$. Similarly, $s \triangleleft \xi = sa_\xi(s)$. Once the global actions of G on M and vice-versa are known, the structure of $G \bowtie M$ is such that

$$su = (s \triangleright u)(u \triangleleft s), \quad \forall u \in G, s \in M. \quad (3)$$

This allows every element of the double cross product group $G \bowtie M$ to be uniquely factorised either as GM or as MG , and relates the two factorisations.

2 T-Duality based on Lie bialgebras

We begin by giving a version of the standard T-duality based on the Drinfeld double of a Lie bialgebra [1][2]. We will phrase it slightly differently in terms of double cross products with a view to later generalisation. Thus, there is a double cross product group $D = G \bowtie M$ with Lie algebra $\mathfrak{d} = \mathfrak{g} + \mathfrak{m}$, and an adjoint-invariant bilinear form on \mathfrak{d} which is zero on restriction to \mathfrak{g} and \mathfrak{m} . The Lie algebra \mathfrak{d} is the direct sum of two perpendicular subspaces \mathcal{E}_- and \mathcal{E}_+ . This means that $\mathfrak{m} = \mathfrak{g}^{*op}$, that the factorisation is a coadjoint matched pair and that $\mathfrak{d} = D(\mathfrak{g})$, the Drinfeld double of \mathfrak{g} , which is the setting that Klimčík etc., assume.

On \mathbb{R}^2 we use light cone coordinates $x_+ = t + x$ and $x_- = t - x$, where t and x are the standard time-space coordinates. Now let us suppose that there is a function $k : \mathbb{R}^2 \rightarrow G \bowtie M$, with the properties that $k_+ k^{-1}(x_+, x_-) \in \mathcal{E}_-$ and $k_- k^{-1}(x_+, x_-) \in \mathcal{E}_+$ for all $(x_+, x_-) \in \mathbb{R}^2$. Then we see that, if we factor $k = us$ for $u \in G$ and $s \in M$,

$$u^{-1}u_\pm + s_\pm s^{-1} \in u^{-1}\mathcal{E}_\mp u.$$

If the projection $\pi_{\mathfrak{g}} : \mathfrak{d} \rightarrow \mathfrak{g}$ (with kernel \mathfrak{m}) is 1-1 and onto when restricted to $u^{-1}\mathcal{E}_-u$ and

$u^{-1}\mathcal{E}_+u$, we can find graph coordinates $E_u : \mathfrak{g} \rightarrow \mathfrak{m}$ and $T_u : \mathfrak{g} \rightarrow \mathfrak{m}$ so that

$$\{\xi + E_u(\xi) : \xi \in \mathfrak{g}\} = u^{-1}\mathcal{E}_+u \quad \text{and} \quad \{\xi + T_u(\xi) : \xi \in \mathfrak{g}\} = u^{-1}\mathcal{E}_-u .$$

It follows that $s_-s^{-1} = E_u(u^{-1}u_-)$ and $s_+s^{-1} = T_u(u^{-1}u_+)$. From the identity

$$(s_+s^{-1})_- - (s_-s^{-1})_+ = [s_-s^{-1}, s_+s^{-1}]$$

we deduce that $u(x_+, x_-)$ satisfies the equation

$$(T_u(u^{-1}u_+))_- - (E_u(u^{-1}u_-))_+ = [E_u(u^{-1}u_-), T_u(u^{-1}u_+)] . \quad (4)$$

Klimčík shows that the Lagrangian density

$$\mathcal{L} = \langle E_u(u^{-1}u_-), u^{-1}u_+ \rangle \quad (5)$$

gives rise to these equations of motion.

The dual theory is given by the factorisation $k = tv$, where $t \in M$ and $v \in G$. If we let $\hat{E}_t : \mathfrak{m} \rightarrow \mathfrak{g}$ and $\hat{T}_t : \mathfrak{m} \rightarrow \mathfrak{g}$ be the graph coordinates of $t^{-1}\mathcal{E}_+t$ and $t^{-1}\mathcal{E}_-t$ respectively, then $t(x_+, x_-)$ obeys the dual equation

$$(\hat{T}_t(t^{-1}t_+))_- - (\hat{E}_t(t^{-1}t_-))_+ = [\hat{E}_t(t^{-1}t_-), \hat{T}_t(t^{-1}t_+)] . \quad (6)$$

These are the equations of motion for a sigma model with Lagrangian

$$\hat{\mathcal{L}} = \langle \hat{E}_t(t^{-1}t_-), t^{-1}t_+ \rangle . \quad (7)$$

These two models are different but equivalent descriptions of the model defined by k . The (u, s) and (t, v) coordinates are related by the actions of the double cross product group structure:

$$tv = (t \triangleright v)(t \triangleleft v) = us. \quad (8)$$

3 Hamiltonian formulation of T-duality

There are two models considered in the last section, the first order equations of motion for $k : \mathbb{R}^2 \rightarrow G \bowtie M$ and the second order equations of motion for $u : \mathbb{R}^2 \rightarrow G$. The equations of motion for $k : \mathbb{R}^2 \rightarrow G \bowtie M$ are the natural way to introduce duality into the system, and are very nearly equivalent to the equations of motion for $u : \mathbb{R}^2 \rightarrow G$. There is not a 1-1 correspondence

between the systems, as multiplying k on the right by a constant element of M gives rise to exactly the same u . We have a Lagrangian and Hamiltonian for the u equations of motion, and can work out the corresponding Hamiltonian mechanics. However the reader must remember that this will not give the Hamiltonian mechanics for k , but rather for k quotiented on the right by constant elements of M .

As pointed out by Klimčík, we can take the phase space of the system to be the set of smooth functions $C^\infty(\mathbb{R}, D)$ (or more strictly $C^\infty(\mathbb{R}, D)/M$), where we regard \mathbb{R} to be a constant time line in \mathbb{R}^{1+1} , or $C^\infty((0, \pi), D)/M$ for a finite space. We will compute the symplectic structure more explicitly than we have found elsewhere and then obtain a new and more symmetric formulation of the Hamiltonian density that covers both the model and the dual model simultaneously. We will need this in later sections when we generalise to arbitrary factorisations, as well as for the point-like limit.

3.1 The symplectic form

We begin by showing that this is the correct phase space, i.e. that such a function encodes both u and \dot{u} on a constant time line. Thus, take $k \in C^\infty(\mathbb{R}, D)$ or $C^\infty((0, \pi), D)$. As $k(x) \in D$ we can factor it as $k(x) = u(x)s(x)$, so $u(x)$ is specified on the constant time line. But we also know that

$$s_x s^{-1} = T_u(u^{-1}u_+) - E_u(u^{-1}u_-) = \frac{1}{2} \left(T_u(u^{-1}\dot{u}) - E_u(u^{-1}\dot{u}) + T_u(u^{-1}u_x) + E_u(u^{-1}u_x) \right), \quad (9)$$

and as we know $s_x s^{-1}$ and $(T_u + E_u)(u^{-1}u_x)$, we can find $(T_u - E_u)(u^{-1}u_t)$. From this we can in principle find $u^{-1}\dot{u}$ as the function $\xi \mapsto T_u(\xi) - E_u(\xi)$ is 1-1 (if η lay in the kernel of this operator then $\eta + T_u(\eta) = \eta + E_u(\eta) \in u^{-1}(\mathcal{E}_+ \cap \mathcal{E}_-)u = \{0\}$).

If we have a system with coordinates for configuration space q_i , and Lagrangian $L(q_i, \dot{q}_i)$, then the canonical momenta are $p_i = \partial L / \partial \dot{q}_i$, and we define a symplectic form on the phase space by $\omega = \sum dp_i \wedge dq_i$. With a little thought, it can be seen that this corresponds to the directional derivative formula (where we have taken a Lagrangian density \mathcal{L})

$$\omega(u, \dot{u}; a, b; c, d) = \int_{x=0}^{\pi} \left(\mathcal{L}''(u, \dot{u}; 0, c; a, b) - \mathcal{L}''(u, \dot{u}; 0, a; c, d) \right) dx.$$

If we write a change in k as labelled by y we get $k_y = u_y s + u s_y$, and likewise for $k_z = u_z s + u s_z$.

From the last section, we can write the Lagrangian density for our system as

$$4\mathcal{L}(u, \dot{u}) = \langle E_u(u^{-1}\dot{u} - u^{-1}u_x), u^{-1}\dot{u} + u^{-1}u_x \rangle ,$$

so we can calculate a partial derivative

$$\begin{aligned} 4\mathcal{L}'(u, \dot{u}; 0, c) &= \langle E_u(u^{-1}c), u^{-1}\dot{u} + u^{-1}u_x \rangle + \langle E_u(u^{-1}\dot{u} - u^{-1}u_x), u^{-1}c \rangle \\ &= \langle E_u(u^{-1}\dot{u}) - T_u(u^{-1}\dot{u}) - E_u(u^{-1}u_x) - T_u(u^{-1}u_x), u^{-1}c \rangle , \end{aligned}$$

so $2\mathcal{L}'(u, \dot{u}; 0, u_y) = -\langle s_x s^{-1}, u^{-1}u_y \rangle$, which results in

$$\begin{aligned} 2\mathcal{L}''(u, \dot{u}; 0, u_y; u_z, \dot{u}_z) &= -\langle (s_x s^{-1})_z, u^{-1}u_y \rangle + \langle s_x s^{-1}, u^{-1}u_z u^{-1}u_y \rangle \\ &= -\langle (s_z s^{-1})_x, u^{-1}u_y \rangle + \langle [s_x s^{-1}, s_z s^{-1}], u^{-1}u_y \rangle + \langle s_x s^{-1}, u^{-1}u_z u^{-1}u_y \rangle \end{aligned}$$

Now compare this with the standard 2-form on the loop group of D . Consider

$$\begin{aligned} \langle (k^{-1}k_y)_x, k^{-1}k_z \rangle &= \langle (s^{-1}s_y)_x + [s^{-1}u^{-1}u_y s, s^{-1}s_x] + s^{-1}(u^{-1}u_y)_x s, s^{-1}s_z + s^{-1}u^{-1}u_z s \rangle \\ &= \langle (s_y s^{-1})_x - [s_x s^{-1}, s_y s^{-1}], u^{-1}u_z \rangle + \langle [s_x s^{-1}, s_z s^{-1}], u^{-1}u_y \rangle \\ &\quad + \langle s_x s^{-1}, [u^{-1}u_z, u^{-1}u_y] \rangle + \langle s_z s^{-1}, (u^{-1}u_y)_x \rangle . \end{aligned}$$

On integration we find

$$\left[\langle s_z s^{-1}, u^{-1}u_y \rangle \right]_{x=0}^{\pi} = \int_{x=0}^{\pi} \left(\langle (s_z s^{-1})_x, u^{-1}u_y \rangle + \langle s_z s^{-1}, (u^{-1}u_y)_x \rangle \right) dx ,$$

so we have the following symplectic form on the phase space:

$$2\omega(k; k_z, k_y) = \int_{x=0}^{\pi} \langle (k^{-1}k_y)_x, k^{-1}k_z \rangle dx - \left[\langle s_z s^{-1}, u^{-1}u_y \rangle \right]_{x=0}^{\pi} . \quad (10)$$

Now we come to the complication, the fact that this form is degenerate on $C^\infty((0, \pi), D)$. If we take a change in $k \in C^\infty((0, \pi), D)$ given by $k\phi$ for $\phi \in \mathfrak{m}$, then $\omega(k; k_z, k\phi) = 0$ for all k_z . To remedy this we could remove the null direction by declaring that the phase space would actually be $C^\infty((0, \pi), D)/M$. Equivalently we could consider the phase space to consist of those $k = us \in C^\infty((0, \pi), D)$ for which $s(0)$ is the identity in M .

3.2 The Hamiltonian density

The Hamiltonian density generating the time evolution can be calculated by

$$4\mathcal{H} = 4\mathcal{L}'(u, \dot{u}; 0, \dot{u}) - 4\mathcal{L}(u, \dot{u}) ,$$

and using our previous result we can write this as

$$\begin{aligned}
4\mathcal{H} &= -\langle E_u(u^{-1}\dot{u} - u^{-1}u_x), u^{-1}u_x \rangle - \langle s_x s^{-1} + E_u(u^{-1}\dot{u} - u^{-1}u_x), u^{-1}\dot{u} \rangle \\
&= -\langle E_u(u^{-1}\dot{u} - u^{-1}u_x), u^{-1}u_x \rangle - \langle T_u(u^{-1}\dot{u}) + T_u(u^{-1}u_x), u^{-1}\dot{u} \rangle \\
&= \langle E_u(u^{-1}u_x), u^{-1}u_x \rangle - \langle T_u(u^{-1}\dot{u}), u^{-1}\dot{u} \rangle = \langle E_u(u^{-1}u_x), u^{-1}u_x \rangle + \langle E_u(u^{-1}\dot{u}), u^{-1}\dot{u} \rangle ,
\end{aligned}$$

or equivalently

$$8\mathcal{H} = \langle (E_u - T_u)(u^{-1}u_x), u^{-1}u_x \rangle + \langle (E_u - T_u)(u^{-1}\dot{u}), u^{-1}\dot{u} \rangle . \quad (11)$$

Using the equation we derived for $s_x s^{-1}$, we can rewrite $\langle (E_u - T_u)(u^{-1}\dot{u}), u^{-1}\dot{u} \rangle$ as

$$\begin{aligned}
&\langle (T_u + E_u)(u^{-1}u_x) - 2s_x s^{-1}, (E_u - T_u)^{-1}((T_u + E_u)(u^{-1}u_x) - 2s_x s^{-1}) \rangle \\
&= -\langle (T_u + E_u)(E_u - T_u)^{-1}(T_u + E_u)(u^{-1}u_x), u^{-1}u_x \rangle \\
&\quad - 4\langle s_x s^{-1}, (E_u - T_u)^{-1}(T_u + E_u)(u^{-1}u_x) \rangle + 4\langle s_x s^{-1}, (E_u - T_u)^{-1}(s_x s^{-1}) \rangle
\end{aligned}$$

If we observe that

$$\langle (E_u - T_u)(u^{-1}u_x), u^{-1}u_x \rangle = \langle (E_u - T_u)(E_u - T_u)^{-1}(E_u - T_u)(u^{-1}u_x), u^{-1}u_x \rangle$$

then we can write

$$\begin{aligned}
4\mathcal{H} &= -\langle T_u(E_u - T_u)^{-1}E_u(u^{-1}u_x), u^{-1}u_x \rangle - \langle E_u(E_u - T_u)^{-1}T_u(u^{-1}u_x), u^{-1}u_x \rangle \\
&\quad - 2\langle s_x s^{-1}, (E_u - T_u)^{-1}(T_u + E_u)(u^{-1}u_x) \rangle + 2\langle s_x s^{-1}, (E_u - T_u)^{-1}(s_x s^{-1}) \rangle . \quad (12)
\end{aligned}$$

To simplify this equation we shall first look at the form of the projections to the subspaces $u^{-1}\mathcal{E}_+u$ and $u^{-1}\mathcal{E}_-u$ in terms of the graph coordinates. If we take $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{m}$, we can write

$$\xi + \phi = (w + E_u(w)) + (y + T_u(y)) ,$$

where $w = (E_u - T_u)^{-1}\phi - (E_u - T_u)^{-1}T_u(\xi)$ and $y = (E_u - T_u)^{-1}E_u(\xi) - (E_u - T_u)^{-1}\phi$. Then we can define projections π_{u+} and π_{u-} to $u^{-1}\mathcal{E}_+u$ and $u^{-1}\mathcal{E}_-u$ as

$$\pi_{u+}(\xi + \phi) = w + E_u(w) \quad \text{and} \quad \pi_{u-}(\xi + \phi) = y + T_u(y) .$$

It follows that

$$(\pi_{u+} - \pi_{u-})\xi = -2E_u(E_u - T_u)^{-1}T_u\xi - (E_u - T_u)^{-1}(T_u + E_u)\xi, \quad (13)$$

$$(\pi_{u+} - \pi_{u-})\phi = 2(E_u - T_u)^{-1}\phi + (T_u + E_u)(E_u - T_u)^{-1}\phi. \quad (14)$$

From this we can rewrite the last equation for the Hamiltonian as

$$4\mathcal{H} = \langle (\pi_{u+} - \pi_{u-})(u^{-1}u_x + s_x s^{-1}), u^{-1}u_x + s_x s^{-1} \rangle.$$

This can be further simplified by removing the u dependence from the projections. If π_+ is the projection to \mathcal{E}_+ with kernel \mathcal{E}_- , then $\pi_{u+} = \text{Ad}_{u^{-1}} \circ \pi_+ \circ \text{Ad}_u$, and since the inner product is adjoint invariant we find

$$4\mathcal{H} = \langle (\pi_+ - \pi_-)(u_x u^{-1} + u s_x s^{-1} u^{-1}), u_x u^{-1} + u s_x s^{-1} u^{-1} \rangle \quad (15)$$

or in terms of combined variable on D ,

$$4\mathcal{H} = \langle (\pi_+ - \pi_-)(k_x k^{-1}), k_x k^{-1} \rangle. \quad (16)$$

The equations of motion can similarly be written in terms of k as

$$\dot{k}k^{-1} = (\pi_- - \pi_+)(k_x k^{-1}). \quad (17)$$

3.3 Symmetries of the models

Returning to the equations of motion in the form $k_{\pm}k^{-1} \in \mathcal{E}_{\mp}$, it is clear that

$$k \mapsto kd, \quad d \in D \quad (18)$$

is a global symmetry of the model. This has been discussed in [2]. In addition to this known symmetry we now consider

$$k \mapsto dk, \quad \mathcal{E}_{\pm} \mapsto d\mathcal{E}_{\mp}d^{-1}, \quad d \in D \quad (19)$$

which alters the subspaces \mathcal{E}_{\pm} and hence the model. On our phase space picture, where the different subspaces appear as different Hamiltonians, this left translation in D may not preserve the Hamiltonian for a particular model, but rather takes us from one model to another.

To have a dynamical symmetry of a particular model we can proceed to restrict to left multiplication by those $d \in D$ such that $d\mathcal{E}_{\pm}d^{-1} = \mathcal{E}_{\pm}$. We distinguish two special cases: (1) The subspaces \mathcal{E}_{\pm} are G -invariant, and (2) The subspaces \mathcal{E}_{\pm} are M -invariant. In case (1) we say that the models are *G-invariant*. Then $T_u = T_e$ and $E_u = E_e$ are independent of $u \in G$,

and the models themselves are simpler to work with. The actions of $d \in G$ by left translation in terms of the variables of the model and the dual model are

$$(u, s) \mapsto (du, s), \quad (t, v) \mapsto ((t^{-1} \triangleleft d^{-1})^{-1}, (t^{-1} \triangleright d^{-1})^{-1} v)$$

respectively. To see if the left translation has a moment map, we consider $k_z = \delta k$ for $\delta \in \mathfrak{d}$ in the equation for the symplectic form:

$$2\omega(k; \delta k, k_y) = \int_{x=0}^{\pi} \langle k(k^{-1}k_y)_x k^{-1}, \delta \rangle dx - \left[\langle s_z s^{-1}, u^{-1}u_y \rangle \right]_{x=0}^{\pi}.$$

If $\delta \in \mathfrak{g}$, then $s_z = 0$, so we have the moment map

$$I_{\delta}(k) = -\frac{1}{2} \int \langle k_x k^{-1}, \delta \rangle dx, \quad \delta \in \mathfrak{g}.$$

In terms of the sigma-model on G , this is

$$-4I_{\delta}(u) = \int \langle 2u^{-1}u_x + (T_u - E_u)(u^{-1}\dot{u}) + (T_u + E_u)(u^{-1}u_x), u^{-1}\delta u \rangle dx, \quad \delta \in \mathfrak{g}$$

which is a conserved charge in the G -invariant case. The left translations for $\delta \in \mathfrak{m}$ are not in general given by moment maps.

There are analogous formulae for the dual model and the M -invariant case. We shall return to these symmetries when we have discussed boundary conditions for the models. We shall also study the particular properties of G -invariant models in some detail in later sections.

4 Solutions independent of x

In this section we show that the systems above in the Hamiltonian form have ‘point-like’ limits where the solutions are restricted so that the field u , say, is independent of x . This then becomes a system of a classical particle moving on the group manifold of G . In the dual picture, i.e. in terms of the variable t , the model is far from point-like and instead describes some form of extended object in the manifold M . We obtain the Poisson brackets and the Hamiltonian and we study the symmetries, in particular the G -invariant case. The dual case where t is pointlike and u extended is identical with the roles of G and M interchanged and is therefore omitted except with regard to the study of this case when the model is G -invariant.

4.1 The point-particle Poisson structure

The solutions which have $u(x)$ independent of x are parameterised by initial values of $u \in G$ and $p = s_x s^{-1} \in \mathfrak{m}$. This is because the equation $s_x s^{-1} = (T_u - E_u)(u^{-1}\dot{u})/2$ shows that p is also independent of x . Therefore the effective phase space coordinates are (u, p) rather than the fields $(u(x), s(x))$ in the general case. The symplectic form per unit length is then

$$2\omega(u, p; u_z, p_z, u_y, p_y) = \langle p_y, u^{-1}u_z \rangle - \langle p_z, u^{-1}u_y \rangle + \langle p, [u^{-1}u_z, u^{-1}u_y] \rangle ,$$

which is closed independently of the pairing used. This can also be written as

$$2\omega(u, p; u_z, p_z; u_y, p_y) = \langle (upu^{-1})_y, u_z u^{-1} \rangle - \langle (upu^{-1})_z, u_y u^{-1} \rangle - \langle p, [u^{-1}u_z, u^{-1}u_y] \rangle. \quad (20)$$

We now invert the symplectic form on the phase space $\mathfrak{m} \times G$ to find the Poisson structure.

Define $\omega_0 : (\mathfrak{m} \oplus \mathfrak{g}) \otimes (\mathfrak{m} \oplus \mathfrak{g}) \rightarrow \mathbb{R}$ by

$$2\omega_0(p_y \oplus \xi_y, p_z \oplus \xi_z) = \langle p_y, \xi_z \rangle - \langle p_z, \xi_y \rangle + \langle p, [\xi_z, \xi_y] \rangle , \quad \forall p_y, p_z \in \mathfrak{m}, \xi_y, \xi_z \in \mathfrak{g}$$

Take a basis e_i of \mathfrak{g} and a dual basis e^i of $\mathfrak{m} = \mathfrak{g}^*$ (for $1 \leq i \leq n$). Then we can take a basis of $\mathfrak{m} \oplus \mathfrak{g}$ as $f_i = e^i$ for $1 \leq i \leq n$ and $f_i = e_{i-n}$ for $n+1 \leq i \leq 2n$. Then in this basis,

$$2\omega_0 = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & A \end{pmatrix} \quad \text{and} \quad (2\omega_0)^{-1} = \begin{pmatrix} A & -\text{id} \\ \text{id} & 0 \end{pmatrix}$$

where $A_{ij} = \langle p, [e_i, e_j] \rangle$. The corresponding tensor is

$$\frac{1}{2}\omega_0^{-1} = \sum_{1 \leq i \leq n} (e_i \otimes e^i - e^i \otimes e_i) + \sum_{1 \leq i, j \leq n} \langle p, [e_i, e_j] \rangle e^i \otimes e^j .$$

Now, $\omega(u, p; u\xi_z, p_z; u\xi_y, p_y) = \omega_0(p_z \oplus \xi_z, p_y \oplus \xi_y)$ so its inverse, the corresponding Poisson bivector is given by left translation from ω_0^{-1} ,

$$\gamma(p, u) = 2 \sum_i \tilde{e}_i \otimes e^i - e^i \otimes \tilde{e}_i + 2\delta p \quad (21)$$

where $\tilde{\xi} = u\xi$ is the left-invariant vector field generated by $\xi \in \mathfrak{g}$.

The Poisson bracket itself then can be described simply for functions f, g on G and $\xi, \eta \in \mathfrak{g} = \mathfrak{m}^*$ by

$$\{f, g\} = 0, \quad \{\xi, f\} = -2\tilde{\xi}(f), \quad \{\xi, \eta\} = 2[\xi, \eta]. \quad (22)$$

From this it is clear that we can quantise the system with the Weyl algebra $\mathbb{C}[G] \rtimes U(\mathfrak{g})$ or at the C^* -algebra level $C(G) \rtimes C^*(G)$ where G acts on G by left multiplication.

4.2 The point-particle Hamiltonian

We have shown that $p = s_x s^{-1}$ is independent of x , so s is of the form $s = e^{px} a$, where $a \in M$ is also independent of x . To find the equations of motion we write $k = u e^{px} a$, where $u \in G$ depends only on time, not on x . Then the equation of motion $\dot{k} k^{-1} = (\pi_- - \pi_+) k_x k^{-1}$ gives

$$\dot{u} u^{-1} + u \frac{d}{dt} (e^{px}) e^{-px} u^{-1} + u e^{px} \dot{a} a^{-1} e^{-px} u^{-1} = (\pi_- - \pi_+) (u p u^{-1}) ,$$

which yields, for the case $x = 0$,

$$u^{-1} \dot{u} + \dot{a} a^{-1} = (\pi_{u-} - \pi_{u+}) p ,$$

and taking the first order terms in x gives

$$\dot{p} = [\dot{a} a^{-1}, p] .$$

We can now get rid of the variable a and write the equations of motion in terms of u and p only,

$$u^{-1} \dot{u} = \pi_{\mathfrak{g}}(\pi_{u-} - \pi_{u+}) p , \quad \dot{p} = [\pi_{\mathfrak{m}}(\pi_{u-} - \pi_{u+}) p, p] .$$

In the constant case, the Hamiltonian per unit length (15) restricts to

$$4\mathcal{H} = \langle (\pi_+ - \pi_-)(u p u^{-1}), u p u^{-1} \rangle . \quad (23)$$

We have to check that the restricted Hamiltonian and the restricted symplectic form indeed correspond to these equations of motion, i.e. that the constraint of x -independence commutes with the original Hamiltonian. To do this, it will be convenient to first calculate from the equations of motion

$$\frac{d}{dt} (u p u^{-1}) = u [(\pi_{u-} - \pi_{u+}) p, p] u^{-1} = [(\pi_- - \pi_+)(u p u^{-1}), u p u^{-1}] ,$$

and now we can write

$$\begin{aligned} 2\omega(u, p; u_z, p_z; \dot{u}, \dot{p}) &= \langle [u p u^{-1}, (\pi_+ - \pi_-) u p u^{-1}], u_z u^{-1} \rangle - \langle (u p u^{-1})_z, \dot{u} u^{-1} \rangle - \langle u p u^{-1}, [u_z u^{-1}, \dot{u} u^{-1}] \rangle \\ &= \langle [u_z u^{-1}, u p u^{-1}], (\pi_+ - \pi_-) u p u^{-1} \rangle - \langle (u p u^{-1})_z - [u_z u^{-1}, u p u^{-1}], \dot{u} u^{-1} \rangle \\ &= \langle (u p u^{-1})_z, (\pi_+ - \pi_-) u p u^{-1} \rangle - \langle u p_z u^{-1}, (\pi_+ - \pi_-) u p u^{-1} \rangle - \langle u p_z u^{-1}, \dot{u} u^{-1} \rangle \\ &= \langle (u p u^{-1})_z, (\pi_+ - \pi_-) u p u^{-1} \rangle - \langle p_z, u^{-1} \dot{u} - (\pi_{u-} - \pi_{u+}) p \rangle \\ &= \langle (u p u^{-1})_z, (\pi_+ - \pi_-) u p u^{-1} \rangle + \langle p_z, \dot{a} a^{-1} \rangle = 2\mathcal{H}_z , \end{aligned}$$

where we used at the end the equations of motion again, and then that $\langle p_z, \dot{a}a^{-1} \rangle = 0$ as \mathfrak{m} is isotropic. In terms of graph coordinates, we can write the equations of motion as

$$u^{-1}\dot{u} = -2(E_u - T_u)^{-1}p = 2T_u^{-1}(E_u^{-1} - T_u^{-1})^{-1}E_u^{-1}p \quad (24)$$

$$\dot{p} = -[(E_u + T_u)(E_u - T_u)^{-1}p, p] = [(E_u^{-1} - T_u^{-1})^{-1}(E_u^{-1} + T_u^{-1})p, p] \quad (25)$$

and the Hamiltonian as

$$4\mathcal{H} = \langle (\pi_{u+} - \pi_{u-})p, p \rangle = 2\langle (E_u - T_u)^{-1}p, p \rangle = 2\langle (E_u^{-1} - T_u^{-1})^{-1}E_u^{-1}p, E_u^{-1}p \rangle. \quad (26)$$

There is also a ‘conjugate’ description of the system which we mention briefly here. Although only $s_x s^{-1} = p$ is directly needed for solving the x -independent equations of motion for the u variable, the rest of the degrees of freedom in s are also an auxiliary part of the system from the point of view of the the group D . It turns out that one could equally regard (p, a) as phase space variables and solve the system in terms of them, with u regarded as auxiliary. Then the equations of motion would be

$$\dot{a}a^{-1} = \pi_{\mathfrak{m}}(\pi_{u-} - \pi_{u+})p = -(E_u + T_u)(E_u - T_u)^{-1}p, \quad \dot{p} = [\pi_{\mathfrak{m}}(\pi_{u-} - \pi_{u+})p, p]. \quad (27)$$

If we work with the phase space $\mathfrak{m} \times M = \mathfrak{g}^* \otimes G^*$, we can more easily compare the system with the classical phase space of the bicrossproduct Hopf algebra $U(\mathfrak{g}) \bowtie \mathbb{C}[G^*]$ associated to the same factorisation of D in [3]. In fact both the Poisson structures and the natural Hamiltonians look somewhat different, but the general interpretation as a particle on $M = G^*$ with momentum given by $p \in \mathfrak{g}^*$ is the same.

4.3 Symmetries of the point-particle system

We now consider which of the translation symmetries of the general theory restrict to the x -independent solutions. First of all, the right translation symmetries are not interesting in this case: the right action by M is the identity on our (u, p) coordinates, while the right action by G does not preserve that u is x -independent. On the other hand, the left translation symmetries by $d \in D$ do preserve that u is x -independent. We compute the Hamiltonian functions for these actions. First of all, for an infinitesimal transformation by $\phi \in \mathfrak{m}$ the variations of u, upu^{-1} are

$$u_\phi = \phi \triangleright u, \quad (upu^{-1})_\phi = [\phi, upu^{-1}]$$

and hence (20) yields

$$2\omega(u, p; u_z, p_z; u_\phi, p_\phi) = -\langle (upu^{-1})_z, \phi \rangle$$

for any variation u_z, p_z . Hence the Hamiltonian function generating this flow is

$$I_\phi(u, p) = -\frac{1}{2}\langle upu^{-1}, \phi \rangle = -h\langle u(p \triangleright u^{-1}), \phi \rangle = -\frac{1}{2}\langle ub_p(u^{-1})u^{-1}, \phi \rangle, \quad \forall \phi \in \mathfrak{m}.$$

Similarly, for an infinitesimal left translation generated by $\xi \in \mathfrak{g}$ we have $u_\xi = \xi u$ (the right-invariant vector field generated by ξ) and $p_\xi = 0$. In this case we obtain more simply

$$2\omega(u, p; u_z, p_z; u_\xi, p_\xi) = -\langle (upu^{-1})_z, \xi \rangle$$

or the generating function

$$I_\xi(u, p) = -\frac{1}{2}\langle upu^{-1}, \xi \rangle = -\frac{1}{2}\langle p \triangleleft u^{-1}, \xi \rangle, \quad \forall \xi \in \mathfrak{g}.$$

The two cases can be combined into a single generating function or moment map

$$I_\delta(u, p) = -\frac{1}{2}\langle upu^{-1}, \delta \rangle, \quad \forall \delta \in \mathfrak{d}. \quad (28)$$

In particular, we see that if the model is G -invariant, so that G is a dynamical symmetry, then the projection of upu^{-1} to \mathfrak{m} ,

$$Q_G = p \triangleleft u^{-1} \quad (29)$$

is a constant of motion, the conserved charge for the symmetry. Likewise, if the model is M -invariant then the projection of upu^{-1} to \mathfrak{g} ,

$$Q_M = ub_p(u^{-1})u^{-1} \quad (30)$$

is a constant of motion.

The Hamiltonian and the equations of motion also simplify in the G -invariant case, namely (24)-(26) with $E_u = E_e$ and $T_u = T_e$. Writing $U = 2(T_e - E_e)^{-1}$, $V = \frac{1}{2}(E_e + T_e)$, we have

$$u^{-1}\dot{u} = Up, \quad \dot{p} = [VUp, p], \quad 4\mathcal{H} = -\langle Up, p \rangle. \quad (31)$$

Thus, the equations of motion decouple in this case; \dot{p} is a quadratic function of p and $u^{-1}\dot{u}$ is a linear function of p , i.e. can then be obtained (in principle) by integrating $p(t)$.

4.4 The extended system dual to the point-particle limit

The dual model when u is x -independent is described by variables t, v both far from x -independent. The dual constraint is one where t is fixed to be x -independent, in which case the model in our original u, s description is far from x -independent. Rather, it is some form of ‘extended solution’.

We can reverse the order of factorisation $k = ue^{px}a = tv$ to get $t^{-1} = (e^{px}a)^{-1}\triangleleft u^{-1}$ and $v^{-1} = (e^{px}a)^{-1}\triangleright u^{-1}$. Here u, p and a are functions of t only. It can be seen that t has a modified exponential behaviour in x , and that v is a constant acted on by an exponential as a function of x . In particular t will not satisfy the Neumann boundary conditions.

The Hamiltonian can be written as

$$4\mathcal{H} = \langle (\pi_{t+} - \pi_{t-})(t^{-1}t_x + v_x v^{-1}), t^{-1}t_x + v_x v^{-1} \rangle ,$$

where $\pi_{t\pm}$ are the projections to $t^{-1}\mathcal{E}_{\pm}t$. The constraints on the dual system corresponding to constant u are that $t\triangleright v$ and $t_x t^{-1} + t v_x v^{-1} t^{-1}$ are independent of x .

5 More about graph coordinates

In this section we provide some preliminary results on the explicit construction of the graph coordinates of the subspaces $\text{Ad}_{u^{-1}}\mathcal{E}_{\pm}$ in terms of the actions of the groups on the Lie algebras. This is needed, in particular, for the explicit computations for the quasitriangular case in the next section. In fact it will be convenient to consider the inverses of the graph coordinates rather than the graph coordinates themselves, as the formulae are considerably simpler.

Thus, given generic \mathcal{E}_{\pm} , the subspace $\text{Ad}_{u^{-1}}\mathcal{E}_{+}$ contains elements of the form

$$\text{Ad}_{u^{-1}}(E_e^{-1}(\phi) \oplus \phi) = \text{Ad}_{u^{-1}}(E_e^{-1}(\phi) + b_{\phi}(u)) \oplus \phi \triangleleft u = E_u^{-1}(\phi \triangleleft u) \oplus \phi \triangleleft u ,$$

so we deduce that

$$E_u^{-1}(\phi) = \text{Ad}_{u^{-1}}(E_e^{-1}(\phi \triangleleft u^{-1}) + b_{\phi \triangleleft u^{-1}}(u)).$$

We can write this as

$$\bar{E}_u^{-1} \equiv \text{Ad}_u \circ E_u^{-1} \circ ((\) \triangleleft u), \quad \bar{E}_u^{-1} = E_e^{-1} + b(u). \quad (32)$$

Also observe that $(s \triangleleft u^{-1}) \triangleright u = (s \triangleright u^{-1})^{-1}$ for any double cross product group, which implies that $\text{Ad}_{u^{-1}}b_{\phi \triangleleft u^{-1}}(u) = -b_{\phi}(u^{-1})$ (this is part of the cocycle property for b). Hence we can write

equivalently

$$E_u^{-1}(\phi) = \text{Ad}_{u^{-1}}(E_e^{-1}(\phi \triangleleft u^{-1})) - b_\phi(u^{-1}). \quad (33)$$

The same formulae hold for T replacing E .

If we consider the dual model the subspace $\text{Ad}_{t^{-1}}\mathcal{E}_+$ contains elements of the form

$$\text{Ad}_{t^{-1}}(\xi \oplus \hat{E}_e^{-1}(\xi)) = t^{-1} \triangleright \xi \oplus \text{Ad}_{t^{-1}}(\hat{E}_e^{-1}(\xi) + a_\xi(t^{-1})) = t^{-1} \triangleright \xi \oplus \hat{E}_t^{-1}(t^{-1} \triangleright \xi),$$

from which we deduce

$$\hat{\bar{E}}_t^{-1} \equiv \text{Ad}_t \circ \hat{E}_t^{-1} \circ (t^{-1} \triangleright (\cdot)), \quad \hat{\bar{E}}_t^{-1} = \hat{E}_e^{-1} + a(t^{-1}) \quad (34)$$

or equivalently that

$$\hat{E}_t^{-1}(\xi) = \text{Ad}_{t^{-1}}(\hat{E}_e^{-1}(t \triangleright \xi)) - a_\xi(t). \quad (35)$$

Similarly for \hat{T} . Note also that $E_e^{-1}(\phi) + \phi \in \mathcal{E}_+$ for all $\phi \in \mathfrak{m}$ and since this also characterises \hat{E}_e (and similarly for \hat{T}_e), we conclude that

$$\hat{E}_e = E_e^{-1}, \quad \hat{T}_e = T_e^{-1}. \quad (36)$$

Finally, we specialise to the case of a coadjoint matched pair, i.e. where \mathfrak{g} is a Lie bialgebra and $\mathfrak{m} = \mathfrak{g}^*$, with $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ the Drinfeld double. Now, associated to the Lie bialgebra structure is a Poisson-Lie group structure on G defined by bivector

$$\gamma_G(u) = \Pi(\tilde{u})$$

where $\tilde{\cdot} = R_*$ denotes extension as a left-invariant vector field and $\Pi : G \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the cocycle $\Pi \in Z_{\text{Ad}}^1(G, \mathfrak{g} \otimes \mathfrak{g})$ extending the Lie cobracket $\delta \in \mathbb{Z}_{\text{ad}}^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ (which is the derivative of Π at the group identity). Since the action of \mathfrak{g}^* on \mathfrak{g} in the coadjoint matched pair is just δ viewed by evaluation against the second factor of its output, the cocycle generator b of its corresponding vector fields on G is just $b = \Pi$ in this case. Also observe that we could equally well have defined γ as generated by *right*-invariant vector fields from some Π^R , say. Here

$$\Pi^R(u) = \text{Ad}_{u^{-1}}(\Pi(u)) = -\Pi(u^{-1}),$$

the last equation by the cocycle condition obeyed by Π .

To apply these observations to the above we write operator $E_u^{-1} : \mathfrak{m} \rightarrow \mathfrak{g}$ as evaluation against the *second* factor of elements $E_u^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$ (we use the same symbols when the meaning is clear). Similarly for \hat{E}_t^{-1} . Then

$$E_u^{-1} = \text{Ad}_{u^{-1}}(E_e^{-1}) - \Pi(u^{-1}) = \text{Ad}_{u^{-1}}(E_e^{-1}) + \Pi^R(u) \quad (37)$$

as elements of $\mathfrak{g} \otimes \mathfrak{g}$. Inverting this defines the Lagrangian for our models,

$$\mathcal{L} = \langle E_u(u^{-1}u_-), u^{-1}u_+ \rangle = E_u(u^{-1}u_+, u^{-1}u_-) \quad (38)$$

where in the second expression we view $E_u : \mathfrak{g} \rightarrow \mathfrak{m}$ as evaluation against the second factor of $E_u \in \mathfrak{m} \otimes \mathfrak{m}$. Or in terms of $\bar{E}_u^{-1} = \text{Ad}_u(E_u^{-1}) \in \mathfrak{g} \otimes \mathfrak{g}$, we have

$$\bar{E}_u^{-1} = E_e^{-1} + \Pi(u) \quad (39)$$

and the Lagrangian written equally as

$$\mathcal{L} = \langle \bar{E}_u(u_-u^{-1}), u_+u^{-1} \rangle = \bar{E}_u(u_+u^{-1}, u_-u^{-1}). \quad (40)$$

One or other of these two forms is usually easier to compute.

Similarly, for the dual model we identify $a(t) : \mathfrak{g} \rightarrow \mathfrak{m}$ with evaluation against the first component of $\hat{\Pi}^R$, i.e. $a = -\hat{\Pi}^R$ when the latter is considered as an operator by evaluation against its second factor (a convention that we adopt unless stated otherwise). Then

$$\hat{E}_t^{-1} = \text{Ad}_{t^{-1}}(E_e) + \Pi^R(t), \quad \hat{\hat{E}}_t^{-1} = E_e + \hat{\Pi}(t) \quad (41)$$

and

$$\mathcal{L} = \hat{E}_t(t^{-1}t_+, t^{-1}t_-) = \hat{\hat{E}}_t(t_+t^{-1}, t_-t^{-1}) \quad (42)$$

is the Lagrangian for the dual model.

These results allow us to explicitly construct the graph coordinates and the Lagrangians given a generic splitting of \mathfrak{d} into subspaces \mathcal{E}_{\pm} . The latter are equivalent to specifying E_e^{-1}, T_e^{-1} and these allow us to obtain the general E_u^{-1} etc., from (33) or from (37) etc., in the coadjoint case.

6 Models based on \mathfrak{g} quasitriangular

In this section we define a class of Poisson-Lie dual models based on the double of \mathfrak{g} (the usual setting) but in the special case where \mathfrak{g} is quasitriangular and factorisable. In this case were

are able to obtain much more explicit formulae for the model and the dual model than in the general case.

A Lie bialgebra is quasitriangular if there is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta\xi = \text{ad}_\xi(r)$ and r obeys the classical Yang-Baxter equations

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad (43)$$

and has $2r_+ = r + r_{21}$ ad-invariant. A factorisable quasitriangular Lie bialgebra is one where $2r_+$ viewed as a map $\mathfrak{g}^* \rightarrow \mathfrak{g}$ is invertible. We denote its inverse by K . In standard examples where \mathfrak{g} is simple, K is a multiple of the Killing form viewed as a map.

In this case there is an isomorphism [20][11]

$$\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* \cong \mathfrak{g}_L \blacktriangleright \mathfrak{g}_R, \quad \xi \oplus \phi \mapsto (\xi + r_1(\phi), \xi - r_2(\phi))$$

which also sends the bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{d} to $K_L - K_R$ on the two copies $\mathfrak{g}_L, \mathfrak{g}_R$ of \mathfrak{g} . Here K_L, K_R are two copies of K . Therefore the inverse image of $\mathfrak{g}_L, \mathfrak{g}_R$ define a splitting of \mathfrak{d} into mutually orthogonal subspaces. From the explicit form of the isomorphism in [11] one finds

$$\mathcal{E}_+ = \{\xi - r_1(K(\xi)) + K(\xi)\}, \quad \mathcal{E}_- = \{\xi - r_2(K(\xi)) - K(\xi)\}. \quad (44)$$

These subspaces are not generic, however (the graphs blow up) but they are the model for the construction which follows. In fact one has a two parameter family of models by varying the coefficients of r_1, K in \mathcal{E}_+ , etc., with graph coordinates in the general case. In another degenerate limit of these parameters one has the principal sigma model as well.

6.1 Construction of the quasitriangular models on G

The subspaces \mathcal{E}_\pm defining our model will be constructed by introducing parameters into (44) in such a way as to preserve orthogonality. Equivalently, one may define suitable E_e^{-1}, T_e^{-1} . We then obtain the general graph coordinates by the method of Section 5. In fact we consider the second problem first as it leads to the most elegant choice of ansatz for the E_e^{-1} etc.

Thus, in the case of a quasitriangular Lie bialgebra one has simply

$$\Pi(u) = \text{Ad}_u(r) - r \quad (45)$$

for the cocycle defining its Poisson structure. This defines the Drinfeld-Sklyanin bracket on G when \mathfrak{g} is the standard quasitriangular structure[19] for a simple Lie algebra \mathfrak{g} . These are

also the Poisson brackets of which the associated quantum groups in this case are the quantisations. We refer to [11] for further discussion of these preliminaries. In view of (45) and the results of Section 5, it is then immediate that the graph coordinates for the model on G in the quasitriangular case obey

$$E_u^{-1} = \text{Ad}_{u^{-1}}(E_e^{-1} - r) + r \quad (46)$$

as an element of $\mathfrak{g} \otimes \mathfrak{g}$. This equation, together with a little linear algebra, allows the explicit computation of the graph coordinates for any model based on a quasitriangular Lie bialgebra, given suitable F_e .

Motivated by (44) we now let

$$E_e^{-1} = (\lambda + 1)r + \mu K^{-1}$$

where λ, μ are two complex parameters. For generic values we will indeed be able to invert to obtain graph coordinates E_u, T_u and hence will obtain a model of the type studied in Sections 2,3.

Clearly, from (47), we have

$$E_u^{-1} = \lambda \text{Ad}_{u^{-1}}(r) + r + \mu K^{-1} \quad (47)$$

as solving the equation (46) for all λ, μ . If we denote by $r_2 : \mathfrak{g}^* \rightarrow \mathfrak{g}$ the evaluation against the second factor of $r \in \mathfrak{g} \otimes \mathfrak{g}$ and similarly by r_1 for evaluation against the first factor, we have equivalently, as maps $\mathfrak{m} \rightarrow \mathfrak{g}$,

$$E_e^{-1} = (\lambda + 1)r_2 + \mu K^{-1} = (\lambda + \mu + 1)r_2 + \mu r_1 \quad (48)$$

for our class of models. Similarly,

$$T_e^{-1} = -(\lambda + 1)r_1 - \mu K^{-1} = -(\lambda + \mu + 1)r_1 - \mu r_2. \quad (49)$$

These imply

$$E_e^{-1} - T_e^{-1} = (\lambda + 1 + 2\mu)K^{-1}, \quad E_e^{-1} + T_e^{-1} = (\lambda + 1)(r_2 - r_1). \quad (50)$$

For further computations in the Hamiltonian formulation we need the difference of the associated projectors π_{\pm} . Rearranging (13)–(14), we have

$$(\pi_{u+} - \pi_{u-})\xi = 2(E_u^{-1} - T_u^{-1})^{-1}\xi + (E_u^{-1} + T_u^{-1})(E_u^{-1} - T_u^{-1})^{-1}\xi, \quad \forall \xi \in \mathfrak{g}, \quad (51)$$

$$(\pi_{u+} - \pi_{u-})\phi = -2E_u^{-1}(E_u^{-1} - T_u^{-1})^{-1}T_u^{-1}\phi - (E_u^{-1} - T_u^{-1})^{-1}(E_u^{-1} + T_u^{-1})\phi, \quad \forall \phi \in \mathfrak{m}. \quad (52)$$

Evaluating at the identity and inserting the above results for E_e^{-1} , etc., we obtain:

$$\langle (\pi_+ - \pi_-)\xi, \xi \rangle = \frac{2}{\lambda + 1 + 2\mu}K(\xi, \xi), \quad \langle (\pi_+ - \pi_-)\xi, \phi \rangle = \frac{\lambda + 1}{\lambda + 1 + 2\mu}K(\xi, (r_1 - r_2)\phi) \quad (53)$$

$$\langle (\pi_+ - \pi_-)\phi, \phi \rangle = \frac{2}{\lambda + 1 + 2\mu}K(T_e^{-1}\phi, T_e^{-1}\phi) \quad (54)$$

$$K(T_e^{-1}\phi, T_e^{-1}\phi) = \frac{(\lambda + 1)^2}{4}K((r_1 - r_2)\phi, (r_1 - r_2)\phi) + \frac{(\lambda + 1 + 2\mu)^2}{4}K^{-1}(\phi, \phi). \quad (55)$$

These results provide for the computation of Hamiltonian from (15) in Section 3.

It remains to show that the above E_e^{-1}, T_e^{-1} indeed define an orthogonal splitting of \mathfrak{d} into subspaces \mathcal{E}_{\pm} and to give these explicitly. First of all the corresponding subspaces defined by our choice of E_e^{-1}, T_e^{-1} are

$$\mathcal{E}_+ = \{E_e^{-1}\phi \oplus \phi\} = \{\xi - \frac{(\lambda + 1)r_1(K(\xi)) - K(\xi)}{\lambda + 1 + \mu} : \xi \in \mathfrak{g}\}, \quad (56)$$

$$\mathcal{E}_- = \{T_e^{-1}\phi \oplus \phi\} = \{\xi - \frac{(\lambda + 1)r_2(K(\xi)) + K(\xi)}{\lambda + 1 + \mu} : \xi \in \mathfrak{g}\}. \quad (57)$$

To show that these form an orthogonal decomposition of \mathfrak{d} , we calculate the inner products

$$\langle E_e^{-1}\phi \oplus \phi, T_e^{-1}\phi \oplus \phi \rangle = \langle E_e^{-1}\phi, \phi \rangle + \langle \phi, T_e^{-1}\phi \rangle = (\lambda + 1)\langle (r_2 - r_1)(\phi), \phi \rangle = 0,$$

$$\langle E_e^{-1}\phi \oplus \phi, E_e^{-1}\phi \oplus \phi \rangle = \langle E_e^{-1}\phi, \phi \rangle + \langle \phi, E_e^{-1}\phi \rangle = (\lambda + 1 + 2\mu)K^{-1}(\phi, \phi),$$

$$\langle T_e^{-1}\phi \oplus \phi, T_e^{-1}\phi \oplus \phi \rangle = \langle T_e^{-1}\phi, \phi \rangle + \langle \phi, T_e^{-1}\phi \rangle = -(\lambda + 1 + 2\mu)K^{-1}(\phi, \phi).$$

In particular, \mathcal{E}_{\pm} are mutually orthogonal as required (the latter two equations show further that the inner product is nondegenerate on each subspace). To show that the subspaces span \mathfrak{d} we need to show that

$$\xi \oplus \phi = E_e^{-1}(\psi) + \psi + T_e^{-1}(\chi) + \chi$$

has a (unique) solution for $\psi, \chi \in \mathfrak{m}$ for all $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{m}$. Clearly $\psi + \chi = \phi$. Meanwhile, putting in the form of E_e^{-1}, T_e^{-1} we have

$$\xi = \mu K^{-1}(\psi - \chi) + (\lambda + 1)(r_2(\psi) - r_1(\chi))$$

which can be rearranged as

$$\xi + (\lambda + 1)(-r_2 + \frac{1}{2}K^{-1})(\phi) = \frac{1}{2}(\lambda + 1 + 2\mu)K^{-1}(\psi - \chi).$$

Thus we have an orthogonal splitting if and only if

$$\lambda + 1 + 2\mu \neq 0. \quad (58)$$

We assume this throughout. Moreover, the splitting has the inverse-graph coordinates E_e^{-1}, T_e^{-1} computed above.

This completes the construction of our model at least in the Hamiltonian formulation. Indeed, this can be defined entirely in terms of E_u^{-1}, T_u^{-1} without recourse to E_u, T_u themselves. It is clear from our construction that:

- (1) The model is G -invariant if and only if

$$\lambda = 0 \quad (59)$$

(or the Lie bialgebra structure on \mathfrak{g} is identically zero.)

- (2) The standard Lagrangian for the model (which requires E_u) exists if and only if (47) are nondegenerate, in particular when μK dominates, i.e.

$$|\mu| \gg |\lambda + 1| \quad (60)$$

and \mathfrak{g} is semisimple.

We describe several special cases.

Modified principal sigma model.

This is obtained by $\lambda = -1, \mu = 1$. Then

$$\mathcal{E}_\pm = \{\xi \pm K(\xi) : \xi \in \mathfrak{g}\}, \quad E_e^{-1} = K^{-1} = -T_e^{-1} \quad (61)$$

Here E_u is obtained by inverting $F_u = K^{-1} - \Pi(u)$ and is not independent of $u \in G$. Considering K, Π as maps K, Π_2 by evaluation against the second component, we have

$$E_u^{-1} - T_u^{-1} = 2K^{-1}, \quad E_u^{-1} + T_u^{-1} = 2\Pi^R(u)$$

for this model. Here $\Pi^R(u)$ defines the Poisson-bracket associated to the Lie bialgebra structure of G and is viewed as a map $\mathfrak{m} \rightarrow \mathfrak{g}$ by evaluation (as usual) against its second factor. In particular, the Lagrangian is

$$\mathcal{L} = \langle (K^{-1} + \Pi^R(u))^{-1} u^{-1} u_-, u^{-1} u_+ \rangle = \langle (K^{-1} + \Pi(u))^{-1} (u_- u^{-1}), u_+ u^{-1} \rangle. \quad (62)$$

This recovers the setting of [2], for example, as a special case of our class of models. Note that the formulae for general μ but $\lambda = -1$ are strictly similar, with $E_u = (\mu K^{-1} + \Pi^R(u))^{-1}$ in the Lagrangian instead.

Pure-quasitriangular and principal sigma model.

The G -invariant models are obtained by $\lambda = 0$, $\mu = 0$. In this case

$$E_u^{-1} = E_e^{-1} = r_2 + \mu K^{-1}, \quad T_u^{-1} = T_e^{-1} = -r_1 - \mu K^{-1}.$$

For the equations of motion we can use the equations $u^{-1}u_- = E_e^{-1}(s_-s^{-1})$ and $u^{-1}u_+ = T_e^{-1}(s_+s^{-1})$ since the operators E_e^{-1} and T_e^{-1} are defined as above, even though E_e and T_e may not be. Then the equations of motion are most conveniently described as a sigma model for s , with equation

$$(T_e^{-1}(s_+s^{-1}))_- - (E_e^{-1}(s_-s^{-1}))_+ = -[E_e^{-1}(s_-s^{-1}), T_e^{-1}(s_+s^{-1})].$$

We see that this case contains another sigma model on the dual group which makes sense in the G -invariant case. Indeed, in the general G -invariant case the variable s may be considered to have a complex parameter μ , which makes this look very much like inverse scattering for the sigma model. Moreover, for generic μ , the operators E_e and T_e do exist, and both u and s are described by sigma models.

The pure-quasitriangular model is the special case with $\mu = 0$ as well. In this case the subspaces \mathcal{E}_\pm are the ones in (44) corresponding to the Drinfeld double as $\mathfrak{g} \blacktriangleright \mathfrak{g}$. This new class of models has Hamiltonian defined by

$$\frac{1}{2} \langle (\pi_+ - \pi_-)(\xi \oplus \phi), \xi \oplus \phi \rangle = K(\xi, \xi) + K(\xi, (r_1 - r_2)\phi) + K(r_1(\phi), r_1(\phi)).$$

The principal sigma model is the limit with $\mu \rightarrow \infty$ and a suitable rescaling. It is on the boundary of our moduli space of quasitriangular models. Then

$$\mathcal{E}_+ = \{\xi + \mu^{-1}(\xi - r_1 \circ K(\xi)) + K(\xi)\}, \quad \mathcal{E}_- = \{\xi + \mu^{-1}(\xi - r_2 \circ K(\xi)) - K(\xi)\} \quad (63)$$

and

$$E_e = \mu^{-1}(K^{-1} + \mu^{-1}r_2)^{-1} = \mu^{-1}K(1 - \mu^{-1}r_2 \circ K + \dots)$$

Hence the Lagrangian is

$$\mathcal{L}(u) = \langle (\mu K^{-1} + r_2)^{-1}(u^{-1}u_-), u^{-1}u_+ \rangle = \mu^{-1}K(u^{-1}u_-, u^{-1}u_+) + \mu^{-2}K(r_2 \circ K(u^{-1}u_-), u^{-1}u_+) + \dots \quad (64)$$

which after an infinite renormalisation has leading term the usual principal sigma model.

The equation of motion, to lowest order in μ^{-1} , is

$$K((u^{-1}u_+)_- + (u^{-1}u_-)_+) = \mu^{-1}(K(r_1 K(u^{-1}u_+)_- + r_2 K(u^{-1}u_1)_+) - [K(u^{-1}u_-), K(u^{-1}u_+)]) + \dots$$

This is the usual principal sigma model equations of motion to lowest order in μ^{-1} , namely

$$(u^{-1}u_+)_- + (u^{-1}u_-)_+ = 0.$$

6.2 Quasitriangular models on SU_2 .

We now compute these models for the group $G = SU_2$ and for its other real form $G = SL_2(\mathbb{R})$.

Actually, only the second of these is strictly real and quasitriangular. Thus, with a basis $\{H, X_{\pm}\}$ for its Lie algebra (with the usual relations), we take the Drinfeld-Sklyanin quasitriangular structure

$$r = X_+ \otimes X_- + \frac{1}{4}H \otimes H.$$

Let $sl_2(\mathbb{R})^*$ have the dual basis $\{\phi, \psi_{\pm}\}$ then its Lie algebra structure is

$$[\phi, \psi_{\pm}] = \frac{1}{2}\psi_{\pm}, \quad [\psi_+, \psi_-] = 0$$

and the other required maps are

$$r_2 \begin{pmatrix} \phi \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{4}H \\ 0 \\ X_+ \end{pmatrix}, \quad K \begin{pmatrix} H \\ X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} 2\phi \\ \psi_- \\ \psi_+ \end{pmatrix}.$$

Note that if we take a different real form

$$e_1 = \frac{-\imath}{2}(X_+ + X_-), \quad e_2 = \frac{-1}{2}(X_+ - X_-), \quad e_3 = \frac{-\imath}{2}H$$

then $[e_i, e_j] = \epsilon_{ijk}e_k$ (the real form su_2) but

$$r = - \sum_i e_i \otimes e_i + \imath(e_1 \otimes e_2 - e_2 \otimes e_1)$$

is not real in this basis. If $\{f_i\}$ is a dual basis then

$$r_2(f_j) = -e_j + \imath e_i \epsilon_{ij3}, \quad K = -\frac{1}{2}\text{id}.$$

This means that although we can arrange for a completely real Lie bialgebra su_2 in this basis (here the Lie coalgebra is purely imaginary but we can rescale r to make it real) it is not a quasitriangular one over \mathbb{R} ; the required r if we want to obey (43) lives in the complexification. In the above conventions the Lie algebra sl_2^* in the dual basis is imaginary,

$$[f_i, f_j] = \imath(\delta_{ik}\delta_{j3} - \delta_{jk}\delta_{i3})f_k.$$

The choice of basis $e_i^* = -\imath f_i$ is its real form su_2^* .

Modified principal sigma model on SU_2 .

To construct the model we will need $\Pi(u) = \text{Ad}_u(r_-) - r_-$ quite explicitly, where $r_- = \imath e_1 \wedge e_2$ is the antisymmetric part of r . For our purposes we write SU_2 as elements

$$u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1.$$

Then working with the matrix representation $e_i = \frac{-\imath}{2}\sigma_i$ given by the Pauli matrices it is easy to find

$$\text{Ad}_{u^{-1}}(e_1) = \Re(a^2 - b^2)e_1 + \Im(a^2 + b^2)e_2 - 2\Re(ab)e_3$$

$$\text{Ad}_{u^{-1}}(e_2) = -\Im(a^2 - b^2)e_1 + \Re(a^2 + b^2)e_2 + 2\Im(ab)e_3$$

and hence

$$\Pi^R(u) = 2\imath e_1 \wedge e_2 |b|^2 - e_3 \wedge e_1 (a\bar{b} - \bar{a}b) - \imath e_2 \wedge e_3 (a\bar{b} + \bar{a}b) \quad (65)$$

after a short computation, which is purely imaginary (as expected). Evaluating against the second factor and regarding as a matrix we have

$$E_u^{-1} = K^{-1} + \Pi^R(u) = -2 \begin{pmatrix} 1 & -\imath|b|^2 & -\imath\Im(a\bar{b}) \\ \imath|b|^2 & 1 & \imath\Re(a\bar{b}) \\ \imath\Im(a\bar{b}) & -\imath\Re(a\bar{b}) & 1 \end{pmatrix}.$$

Here $E_u^{-1}(f_j) = E_{ij}^{-1}e_i$, where (E_{ij}^{-1}) is the matrix shown. Note that we can write

$$E_{ij}^{-1} = -2(\delta_{ij} + \imath\epsilon_{ijk}\pi_k), \quad \pi = \begin{pmatrix} \Re(a\bar{b}) \\ \Im(a\bar{b}) \\ -|b|^2 \end{pmatrix}$$

and any matrix of this form has inverse

$$E_{ij} = -\frac{1}{2(1 - \pi^2)}(\delta_{ij} - \imath\epsilon_{ijk}\pi_k - \pi_i\pi_j).$$

Here $\pi^2 = \pi \cdot \pi = |b|^2$ in our case. The corresponding operator is $E_u(e_j) = E_{ij}f_i$. To cast the resulting Lagrangian in a useful form let us note that

$$\text{Tr}(\text{id} - \not{\pi})\sigma_i\sigma_j = \text{Tr}(\text{id} - \pi \cdot \sigma)(\delta_{ij}\text{id} + \imath\epsilon_{ijk}\sigma_k) = 2(\delta_{ij} - \imath\epsilon_{ijk}\pi_k)$$

where σ_i are the Pauli matrices and $\not{\pi} = \pi \cdot \sigma$. Hence in our representation of su_2 in basis $e_i = \frac{-\imath\sigma_i}{2}$ we have

$$\mathcal{L} = \frac{1}{|a|^2} \left(\text{Tr}[(\text{id} - \not{\pi})u^{-1}u_+u^{-1}u_-] - \frac{1}{2}\text{Tr}[\not{\pi}u^{-1}u_+]\text{Tr}[\not{\pi}u^{-1}u_-] \right) \quad (66)$$

where

$$\not{\pi} = \begin{pmatrix} -|b|^2 & \bar{a}b \\ a\bar{b} & |b|^2 \end{pmatrix} = \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} u = u^{-1} \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}.$$

The matrix E_{ij} here is complex since Π^R in our conventions is imaginary. For a completely real version of this model on SU_2 one should keep the freedom of general μ in this class of models so that $E_u^{-1} = \mu K^{-1} + \Pi^R(u)$ and then set $\mu = \imath$. Taking the real normalisation of su_2 as a Lie bialgebra (i.e. multiplying r by $-\imath$ so that $r_- = e_1 \wedge e_2$ and $K^{-1} = 2\text{id}$) gives the same E_{ij}^{-1} as above but times $-\imath$ off the diagonal. One may also work of course on $G = SL_2(\mathbb{R})$ with real r, K for a completely real model with $\mu = 1$.

This class of models has been considered specifically for SU_2 in [13], although not so explicitly as above.

Pure-quasitriangular and principal sigma models on SU_2 .

Here we take $\lambda = 0$ and can write down immediately

$$E_u^{-1} = E_e^{-1} = - \begin{pmatrix} 1+2\mu & -\imath & 0 \\ \imath & 1+2\mu & 0 \\ 0 & 0 & 1+2\mu \end{pmatrix}$$

which has inverse

$$E_u = E_e = \frac{-1}{4\mu} \begin{pmatrix} \frac{1+2\mu}{1+\mu} & \frac{\imath}{1+\mu} & 0 \\ \frac{-\imath}{1+\mu} & \frac{1+2\mu}{1+\mu} & 0 \\ 0 & 0 & \frac{4\mu}{1+2\mu} \end{pmatrix}$$

for $\mu \neq 0, -\frac{1}{2}, -1$. The Lagrangian defined by this can be conveniently obtained by writing

$$E_{ij}^{-1} = -(1+2\mu)(\delta_{ij} - \imath\epsilon_{ijk}\pi_k), \quad \pi = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{1+2\mu} \end{pmatrix}$$

which implies (by similar computations to those above),

$$\mathcal{L} = \frac{1}{\mu(1+\mu)} \left(\text{Tr} \left[\begin{pmatrix} 1+\mu & 0 \\ 0 & \mu \end{pmatrix} u^{-1}u_+u^{-1}u_- \right] - \frac{1}{4(1+2\mu)} \text{Tr} [\sigma_3 u^{-1}u_+] \text{Tr} [\sigma_3 u^{-1}u_-] \right).$$

This singular for the pure quasitriangular model where $\mu = 0$, and also does not have a good limit at $\mu = \infty$ for the principal sigma model. Rather, we have well-defined equations of motion conveniently described as a sigma model for $s \in M$ as explained above, using E_e^{-1} and a similar matrix for T_e^{-1} .

On the other hand, by changing the normalisation of the Lie bialgebra structure (namely, dividing r by μ) we have E_u with the same matrix as above but without the μ^{-1} factor in front. This rescaled Lagrangian is well defined both for $\mu = 0$ and $\mu = \infty$, with

$$\mu \mathcal{L} \rightarrow \begin{cases} \frac{1}{2} \text{Tr} [(1 + \sigma_3) u^{-1}u_+u^{-1}u_-] - \frac{1}{4} \text{Tr} [\sigma_3 u^{-1}u_+] \text{Tr} [\sigma_3 u^{-1}u_-] & \text{as } \mu \rightarrow 0 \\ \text{Tr} [u^{-1}u_+u^{-1}u_-] & \text{as } \mu \rightarrow \infty \end{cases}.$$

The first limit is the Lagrangian for the rescaled pure-quasitriangular model on SU_2 , while the second is the standard Lagrangian for the principal sigma model on SU_2 based on the Killing form of su_2 .

Notice that in this rescaled model the Lie cobracket of su_2 is infinite at $\mu = 0$, i.e. the Lie algebra \mathfrak{m} has infinite commutators, and zero at $\mu = \infty$, i.e. the Lie algebra \mathfrak{m} is Abelian. The geometrical pictures behind these two models are therefore very different but interpolated by general μ .

Also note that the $\mu = 0$ limit here is again defined by a complex Lagrangian. For a real version one may look at the pure-quasitriangular model on $G = SL_2(\mathbb{R})$ instead. Here we have, clearly,

$$E_e^{-1} \begin{pmatrix} \phi \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \frac{1+2\mu}{4} H \\ \mu X_- \\ (1+\mu) X_+ \end{pmatrix}, \quad E_e \begin{pmatrix} H \\ X_+ \\ X_- \end{pmatrix} = \mu^{-1} \begin{pmatrix} \frac{4\mu}{1+2\mu} \phi \\ \frac{\mu}{1+\mu} \psi_- \\ \psi_+ \end{pmatrix}.$$

As before, we take out a factor μ by rescaling in order to obtain well-defined operators E_e at $\mu = 0, \infty$, this time with all coefficients being real in our choice of bases. The corresponding Lagrangian can easily be written out explicitly upon fixing a description of $u \in SL_2(\mathbb{R})$. For example, if we write

$$u = e^{xX_+} e^{hH} e^{yX_-}$$

so that

$$u^{-1}u_{\pm} = x_{\pm} X_{\pm} e^{-2h} + (h_{\pm} + y x_{\pm} e^{-2h}) H + (y_{\pm} - 2y h_{\pm} - 2y^2 x_{\pm} e^{-2h}) X_{\mp}$$

using the relations of sl_2 then the rescaled $\mu = 0$ limit gives the Lagrangian

$$\mathcal{L} = e^{-2h} x_+ (y_- - 2yh_- - 2y^2 e^{-2h} x_-)$$

as the pure-quasitriangular model on $SL_2(\mathbb{R})$. The $\mu = \infty$ limit is the standard principal sigma model on $SL_2(\mathbb{R})$ and the general case interpolates the two.

6.3 Dual of the quasitriangular models on G^\star

The quasitriangular models are examples of the case where the factorisation is based on the Drinfeld double associated to a Lie bialgebra, so that E_u^{-1} is related to the Poisson-Lie group G . Hence the dual models are of the same form but based on the Poisson-Lie group G^\star rather than G , i.e. with $\hat{\Pi}(t) \in \mathfrak{m} \otimes \mathfrak{m}$ in place of Π . As explained in Section 5 we can then construct them from the initial data

$$\hat{E}_e^{-1} = E_e, \quad \hat{T}_e^{-1} = T_e$$

as given above for our quasitriangular models. We compute $\hat{E}_t^{-1} = E_e + \hat{\Pi}(t)$ and invert to obtain the Lagrangian

$$\hat{\mathcal{L}} = \langle (E_e + \hat{\Pi}(t))^{-1} t_- t^{-1}, t_+ t^{-1} \rangle \quad (67)$$

for the dual model. For the models below, where there is no special Ad_t -invariance of E_e , this is easier than computing the Lagrangian via \hat{E}_t .

We outline the results for SU_2^\star and $SL_2(\mathbb{R})^\star$. First of all we describe these groups explicitly. The former is generated by the basis $\{-\imath f_i\}$, i.e. we write $\phi = \phi_i(-\imath f_i) \in \mathfrak{m}$ for real ϕ_i , which we regard as a vector $\vec{\phi}$. One standard representation of the resulting group is as matrices of the form

$$\begin{pmatrix} x & z \\ 0 & x^{-1} \end{pmatrix}, \quad x > 0, \quad z \in \mathbb{C}.$$

This is the group occurring in the Iwasawa decomposition $SL_2(\mathbb{C}) = SU_2 \rtimes SU_2^\star$, see [11]. Another description useful for very explicit computations is as the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ [11], which can be viewed as a modified product on \mathbb{R}^3 . Elements are $\vec{s} \in \mathbb{R}^3$ with $s_3 > -1$ and the product law and inversion are

$$\vec{s}\vec{t} = \vec{s} + (s_3 + 1)\vec{t}, \quad \vec{s}^{-1} = -\frac{\vec{s}}{s_3 + 1}.$$

The exponentiation from the Lie algebra to a group is explicitly

$$\vec{s} = \vec{\phi} \frac{e^{\phi_3} - 1}{\phi_3}$$

for $\vec{s} = e^{\vec{\phi}}$ in the natural 3-dimensional coadjoint representation. See [11]. The real form $SL_2(\mathbb{R})^*$ has a similar description as $\mathbb{C} \rtimes \mathbb{R}$, i.e. where s_2 is imaginary and s_1, s_3 real with $s_3 > -1$ according to the conventions in [11]. Note that $x = s_3 + 1$ is multiplicative under the group law if one wants a more standard notation.

The Lie bracket on su_2 determines the Lie cobracket and Poisson structure on SU_2^* (and similarly on $SL_2(\mathbb{R})^*$). It is given by [11]

$$\hat{\Pi}(s) = -\imath(\epsilon_{ija}s_a + \frac{1}{2}s^2\epsilon_{ij3})f_i \otimes f_j.$$

Explicitly,

$$\imath\hat{\Pi}(s) = \frac{1}{2}(s_1^2 + s_2^2 + (s_3 + 1)^2 - 1)f_1 \wedge f_2 + s_2f_3 \wedge f_1 + s_1f_2 \wedge f_3.$$

Note also that the notation $s_{\pm}s^{-1}$ means more precisely $R_{s^{-1}*}s_{\pm}$. Similarly for $s^{-1}s_{\pm}$. In our present group coordinates, from the product law, it is easy to see that

$$L_{s*}\vec{\phi} = (s_3 + 1)\vec{\phi}, \quad R_{s*}\vec{\phi} = \vec{\phi} + \phi_3\vec{s}.$$

Dual of the modified principal sigma model.

We set $\lambda = -1$ and $\mu = 1$. Then

$$E_e = K = -\frac{1}{2} \sum_i f_i \otimes f_i$$

Hence

$$\hat{E}_{ij}^{-1} = -\frac{1}{2}(\delta_{ij} + \imath\epsilon_{ijk}\hat{\pi}_k), \quad \hat{\pi} = 2\vec{t} + \begin{pmatrix} 0 \\ 0 \\ t^2 \end{pmatrix}$$

and

$$\hat{E}_{ij} = -\frac{2}{1 - t^2(t^2 + 4(t_3 + 1))}(\delta_{ij} - \imath\epsilon_{ijk}\hat{\pi}_k - \hat{\pi}_i\hat{\pi}_j)$$

This defines the Lagrangian

$$\hat{\mathcal{L}} = \frac{2}{1 - t^2(t^2 + 4(t_3 + 1))} \left(\nabla_+\vec{t} \cdot \nabla_-\vec{t} - \imath\hat{\pi} \cdot (\nabla_+\vec{t} \times \nabla_-\vec{t}) - (\hat{\pi} \cdot \nabla_+\vec{t})(\hat{\pi} \cdot \nabla_-\vec{t}) \right)$$

where $R_{t^{-1}*}t_{\pm}$ is computed as

$$\nabla_{\pm}\vec{t} = \vec{t}_{\pm} - t_{\pm 3} \frac{\vec{t}}{t_3 + 1}.$$

As before, the model in the form stated is complex but with a different choice $\mu = \imath$ and different normalisation of r we can obtain a real model as well.

Dual of the pure-quasitriangular and principal sigma models.

Here we set $\lambda = 0$. Then rearranging E_e above as an element of $\mathfrak{m} \otimes \mathfrak{m}$ we have

$$E_e = -\frac{1}{4\mu} \left(\frac{1+2\mu}{1+\mu} (f_1 \otimes f_1 + f_2 \otimes f_2) + \frac{4\mu}{1+2\mu} f_3 \otimes f_3 + \frac{\imath}{1+\mu} f_1 \wedge f_2 \right).$$

One may then compute

$$\hat{\bar{E}}_t = (E_e + \hat{\Pi})^{-1}$$

and hence the Lagrangian. The result does not have any particular simplifying features over the $\lambda = -1$ case above, so we omit its detailed form.

Both limits of μ are singular, and require rescaling. The $\mu \rightarrow \infty$ case makes sense after a rescaling of r to r/μ . This in turn scales the Lie cobracket of \mathfrak{g} by μ^{-1} and hence also changes the Lie algebra structure of \mathfrak{m} to an Abelian one plus corrections of order μ^{-1} . The effect of this is to change the exponential map and the group law of G^* , making the latter Abelian. This can be expressed conveniently by working in new coordinates with \vec{t} scaled by μ^{-1} . In this new coordinate system we have

$$\hat{\bar{E}}_t = -2\text{id} + O(\mu^{-1})$$

since $\hat{\Pi}$ is linear in \vec{t} to lowest order. The Lagrangian is

$$\hat{\mathcal{L}} = 2\vec{t}_+ \cdot \vec{t}_- + O(\mu^{-1}).$$

Thus the dual model to the principal sigma model on SU_2 is an Abelian one based on the group \mathbb{R}^3 with the usual linear wave equation.

The similar limit for the pure-quasitriangular case is ill-defined since the Lie bracket of \mathfrak{m} becomes singular as $\mu \rightarrow 0$. Other scaling limits of both the original model and its dual are possible in this case.

6.4 Point-particle limit of the quasitriangular models

We have seen that the point-particle limit where u is independent of x reduces to a classical mechanical dynamical system on the group G . For our quasitriangular models we have the following special cases.

Point-particle modified principal model.

From the expressions for E_u^{-1} etc. above, the Hamiltonian is

$$\mathcal{H} = \frac{1}{4}K^{-1}((K \circ \Pi^R(u) + 1)p, (K \circ \Pi_2(u^{-1}) + 1)p) \quad (68)$$

and the equations of motion are

$$u^{-1}\dot{u} = K^{-1} \circ ((K \circ \Pi^R(u))^2 - 1)p, \quad \dot{p} = [K \circ \Pi^R(u)p, p]. \quad (69)$$

In this limit both the case entirely over \mathbb{R} or the case where r and hence the cobracket are imaginary lead to well-defined real equations of motion. In this case Π is imaginary but so is the Lie bracket of \mathfrak{m} in the dual basis to the real basis of \mathfrak{g} .

For example, we can either work on $G = SL_2(\mathbb{R})$ or, as more usual, on $G = SU_2$. In the latter case (see above) we have

$$K \circ \Pi^R(u) = \imath \begin{pmatrix} 0 & -|b|^2 & -\Im(a\bar{b}) \\ |b|^2 & 0 & \Re(a\bar{b}) \\ \Im(a\bar{b}) & -\Re(a\bar{b}) & 0 \end{pmatrix} = \imath \epsilon_{ijk} \pi_k f_i \otimes e_j.$$

Using the complexified Lie bracket on su_2^* we have the equations of motion for $p = p_i f_i$ (with p_i real) as

$$\dot{\vec{p}} = \vec{p}(\vec{p} \times \pi)_3 - p_3 \vec{p} \times \pi$$

in terms of the vector cross product. This can be written explicitly as

$$\dot{p}_3 = 0, \quad \dot{\rho} = -\frac{\imath}{2}\bar{a}b\rho^2 + \frac{\imath}{2}a\bar{b}(|\rho|^2 + 2p_3^2) + \imath|b|^2\rho p_3, \quad \rho \equiv p_1 + \imath p_2$$

after a short computation. On the other hand,

$$(K \circ \Pi^R)^2 - 1)_{ij} = (\pi^2 - 1)\delta_{ij} - \pi_i \pi_j$$

hence the equation for u is in our basis $e_i = \frac{-\imath\sigma_i}{2}$ of su_2 is

$$u^{-1}\dot{u} = \imath(\pi^2 - 1)\not{p} - \imath\not{\pi} \cdot p.$$

In our case $\pi^2 = |b|^2$ and $\pi \cdot p = \Re(\rho\bar{a}b) - |b|^2 p_3$, hence

$$\dot{u} = -\frac{\imath}{2} \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} (\rho\bar{a}b + \bar{\rho}a\bar{b} - 2|b|^2 p_3) - \imath|a|^2 u \begin{pmatrix} p_3 & \bar{\rho} \\ \rho & -p_3 \end{pmatrix}.$$

Explicitly, this is

$$\dot{a} = -\imath|a|^2(ap_3 + b\rho), \quad \dot{b} = \imath bp_3 - \frac{\imath}{2}(1 + |a|^2)a\bar{\rho} - \frac{\imath}{2}\rho\bar{a}b^2.$$

One may verify that this preserves $|a|^2 + |b|^2 = 1$ as it must.

Point-particle pure-quasitriangular model.

We set $\lambda = 0$ and $E_u = E_e$ etc (the models are G -invariant). The Hamiltonian and equations of motion are then

$$2\mathcal{H} = \frac{1}{1+2\mu} K((r_2 + \mu K^{-1})p, (r_2 + \mu K^{-1})p) \quad (70)$$

$$u^{-1}\dot{u} = -\frac{2}{1+2\mu}(r_2 + \mu K^{-1}) \circ K \circ (r_1 + \mu K^{-1})p, \quad \dot{p} = \frac{2}{1+2\mu}[K \circ r_2 p, p]. \quad (71)$$

Since these models are invariant, we know that $p \triangleleft u^{-1}$ is conserved. This means that we can let $Q = p(0) \triangleleft u(0)^{-1} \in \mathfrak{m}$ be fixed and substitute $p(t) = Q \triangleleft u(t)$ into the equation for \dot{u} . We then solve a first order non-linear differential equation for $u(t)$.

In particular, in the limit $\mu = 0$ we obtain the x -independent limit of the pure-quasitriangular model. Thus

$$\mathcal{H} = \frac{1}{2}K(r_2 p, r_2 p), \quad u^{-1}\dot{u} = 2(r_2 \circ K - 1)r_2 p, \quad \dot{p} = 2[K \circ r_2 p, p] \quad (72)$$

using $r_1 + r_2 = K^{-1}$ to rearrange. In this case it makes sense to consider the reduced variable $\xi = r_2 p$ and write the equations of motion as

$$u^{-1}\dot{u} = 2(r_2 \circ K - 1)\xi, \quad \dot{\xi} = 2[r_2 \circ K \xi, \xi] \quad (73)$$

where we use that $r_2 : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is a Lie algebra homomorphism in view of the classical Yang-Baxter equation (43)[11]. We only need to solve this for ξ in the image of r_2 but it is interesting that the equation makes sense for any ξ as an interesting integrable system on the group manifold.

We can solve this for our strictly real form $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. We will solve it here for the general (73); the special case of interest is similar but more elementary. Thus,

$$r_2 \circ K \begin{pmatrix} H \\ X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} \frac{1}{2}H \\ X_+ \\ 0 \end{pmatrix}$$

so, writing $\xi(t) = h(t)H + x(t)X_+ + y(t)X_-$, we need to solve

$$u^{-1}\dot{u} = -h(t)H - 2y(t)X_-$$

and

$$\dot{h}H + \dot{x}X_+ + \dot{y}X_- = [hH + 2xX_+, hH + xX_+ + yX_-] = 2xyH - 2hxX_+ - 2hyX_-,$$

which is the system of equations

$$\dot{h} = 2xy, \quad \dot{x} = -2hx, \quad \dot{y} = -2hy.$$

Note first of all that

$$\frac{d}{dt}(h^2 + \frac{1}{2}\dot{h}) = 0$$

so

$$h^2 + xy = \frac{\omega^2}{4}$$

(say) is a constant. Inserting this into the equation for h yields the Riccati equation

$$\dot{h} - \frac{1}{2}\omega^2 + 2h^2 = 0$$

which has the general solution

$$h(t) = \frac{1}{2}\omega \frac{\sinh(\omega t) + \frac{2h(0)}{\omega} \cosh(\omega t)}{\cosh(\omega t) + \frac{2h(0)}{\omega} \sinh(\omega t)}.$$

We can then compute y as

$$y(t) = e^{-2 \int_0^t h(\tau) d\tau} y(0)$$

and similarly for $x(t)$. Since we only need h, y to obtain $u(t)$ we can consider the choice of $x(0)$ to be equivalent to the choice of ω (at least in a certain range). The initial values of h, y then determine their general values as above, and these then determine $u(t)$ given $u(0)$. The latter can be expressed explicitly in terms of integrals on fixing a coordinate system for $SL_2(\mathbb{R})$.

For the point-particle limit of the pure quasitriangular models we are only interested in $\xi \in b_+$ (the image of r_2), i.e. we specialise to solutions of the form $y(0) = 0$, which clearly implies $y(t) = 0$ and $\dot{h} = 0$. In this case the solution is clearly

$$\xi(t) = \frac{\omega}{2}H + e^{-\omega t}x(0)X_+, \quad u(t) = u(0)e^{-\frac{1}{2}\omega t H}.$$

for initial data $\omega, x(0), u(0)$.

For the full physical momentum $p(t)$ we go back to (72). If we write $p = 2\omega\phi + x\psi_- + \bar{x}\psi_+$ say, then a similar computation using the Lie algebra of $sl_2(\mathbb{R})^*$ gives ω constant, $\dot{x} = -\omega x$ as before, and additionally $\dot{\bar{x}} = \omega \bar{x}$. Hence the solution is

$$p(t) = 2\omega\phi + e^{-\omega t}x(0)\psi_- + e^{\omega t}\bar{x}(0)\psi_+, \quad u(t) = u(0)e^{-\frac{1}{2}\omega t H}$$

for constants $\omega, x(0), \bar{x}(0)$. As a check, it is easy to verify that

$$Q_G = p \lrcorner u^{-1} = (p \lrcorner e^{\frac{1}{2}\omega t H}) \lrcorner u(0)^{-1}$$

is conserved. Here $\psi_{\pm} \lrcorner H = \mp 2\psi_{\pm}$ and $\phi \lrcorner H = 0$ is the relevant coadjoint action.

Point-particle principal model.

In the limit $\mu \rightarrow \infty$ of (70)–(71), we obtain the x -independent limit of the principal sigma model.

Here

$$4\mathcal{H} = K^{-1}(p, p), \quad u^{-1}\dot{u} = -K^{-1}\bar{p}, \quad \dot{\bar{p}} = 0$$

where $\bar{p} = \mu p$ is the renormalised momentum variable. This has the general solution

$$u(t) = u(0)e^{-tK^{-1}\bar{p}}, \quad \bar{p}(t) = \bar{p}(0).$$

It is easy to see that $Q = \bar{p} \lrcorner u^{-1}$ is constant as well, using K ad-invariant.

7 Generalised T-Duality with double Neumann boundary conditions

So far we have worked on providing a special class of Poisson-Lie T-dual models within the established general framework. We now return to our Hamiltonian formulation of the general framework and observe that in this form the main ideas can be extended to a much more general setting. Thus, from the symplectic form and the Hamiltonian we have just calculated, we can see how the definition of T-duality could be generalised. Begin with a Lie group D , with Lie algebra \mathfrak{d} , and suppose that \mathfrak{d} is the direct sum of two subspaces \mathcal{E}_- and \mathcal{E}_+ . We take π_+ to be the projection to \mathcal{E}_+ with kernel \mathcal{E}_- , and π_- to be the projection to \mathcal{E}_- with kernel \mathcal{E}_+ .

Suppose that there is a function $k : \mathbb{R}^2 \rightarrow D$, with the properties that $k_+k^{-1}(x_+, x_-) \in \mathcal{E}_-$ and $k_-k^{-1}(x_+, x_-) \in \mathcal{E}_+$ for all $(x_+, x_-) \in \mathbb{R}^2$. Then the relation $k_+k^{-1}(x_+, x_-) \in \mathcal{E}_-$ can be summarised by $\pi_+(k_+k^{-1}) = 0$, and similarly we get $\pi_-(k_-k^{-1}) = 0$. This gives the equations of motion on

$$\dot{k}k^{-1} = (\pi_- - \pi_+)(k_x k^{-1}).$$

Now we look at the symplectic form on the phase space. Suppose that \mathfrak{d} has an adjoint invariant inner product $\langle \cdot, \cdot \rangle$. If we imposed boundary conditions that $k(0)$ and $k(\pi)$ were fixed,

then the symplectic form we computed earlier becomes

$$2\omega(k; k_z, k_y) = \int_{x=0}^{\pi} \langle (k^{-1}k_y)_x, k^{-1}k_z \rangle dx .$$

If we substitute $k_z = \dot{t}$, then we get

$$2\omega(k; \dot{k}, k_y) = \int_{x=0}^{\pi} \langle k(k^{-1}k_y)_x k^{-1}, \dot{k}k^{-1} \rangle dx = \int_{x=0}^{\pi} \langle (k_x k^{-1})_y, (\pi_- - \pi_+)(k_x k^{-1}) \rangle dx .$$

and so

$$4\omega(k; \dot{k}, k_y) = -D_{(k; k_y)} \int_{x=0}^{\pi} \langle k_x k^{-1}, (\pi_+ - \pi_-)(k_x k^{-1}) \rangle dx ,$$

on the assumption that $\pi_+ - \pi_-$ is Hermitian. This will be true if the subspaces \mathcal{E}_- and \mathcal{E}_+ are perpendicular with respect to the inner product. Then we see that $\omega(k; k_y, \dot{k}) = D_{(k; k_y)} \mathcal{H}(k)$, where

$$4\mathcal{H} = \langle (\pi_{u_+} - \pi_{u_-})(u^{-1}u_x + s_x s^{-1}), u^{-1}u_x + s_x s^{-1} \rangle .$$

gives the Hamiltonian generating the time evolution.

The form of the boundary conditions we have imposed here should not come as too much of a surprise. Normally the string has boundary conditions (for $k = us$ with $u \in G$ and $t \in M$) $u_x = 0$ at $x = 0$ or $x = \pi$. This Neumann condition is designed to prevent momentum transfer out of the string at the edges. But if the system is to be completely dual, we also need to impose a corresponding Neumann condition on the dual theory, which leads to the boundary condition $k_x = 0$, the ‘double Neumann’ condition. But then the equation of motion states $\dot{k} = 0$ on the boundary. Alternatively, if the reader prefers to work over $x \in \mathbb{R}$, we just deal with rapidly decreasing solutions. In either of these cases, the symplectic form really is non-degenerate.

Now we have a phase space and Hamiltonian for the equations of motion just based on an invariant inner product on D and an orthogonal decomposition \mathcal{E}_- and \mathcal{E}_+ of \mathfrak{d} . If we take D to be a doublecross product $D = G \bowtie M$, and assume that the subspaces $\text{Ad}_{u^{-1}} \mathcal{E}_{\pm}$ have graph coordinates T_u and E_u as before, we again recover the previous equations of motion for $u \in G$ in the factorisation $k = us$,

$$(T_u(u^{-1}u_+))_- - (E_u(u^{-1}u_-))_+ = [E_u(u^{-1}u_-), T_u(u^{-1}u_+)] .$$

Importantly, we do not need to assume that the inner product has any special properties with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{m}$ (such as being zero on \mathfrak{g}). We can also give the form of

the Hamiltonian for this general case:

$$4\mathcal{H} = \langle (E_u + I)(u^{-1}u_-), (E_u + I)(u^{-1}u_-) \rangle - \langle (T_u + I)(u^{-1}u_+), (T_u + I)(u^{-1}u_+) \rangle .$$

The corresponding dual formula would produce exactly the same value.

7.1 Poisson brackets and the central extension

In this section we continue with the generalised T-duality and boundary conditions of the last section. The phase space for our system is infinite dimensional, so it is rather hard to describe the functions on it directly. We shall describe a ‘nice’ set of functions, and hope that more general functions are expressible as a product of these nice functions.

If $v \in C^\infty((0, \pi), \mathfrak{d})$, we can look at the vector field $k_z = vk$ for $k \in C^\infty((0, \pi), D)$. To preserve the boundary conditions we consider only those $v \in C^\infty((0, \pi), \mathfrak{d})$ which tend to zero at the end points. Consider

$$\omega(k; k_y, k_z) = -\frac{1}{2} \int \langle (k^{-1}k_y)_x, k^{-1}k_z \rangle dx = -\frac{1}{2} \int \langle (k_x k^{-1})_y, v \rangle dx = -\frac{1}{2} D_{(k;y)} \int \langle k_x k^{-1}, v \rangle dx .$$

It follows that the function which acts as a Hamiltonian generating this flow is

$$f_v(k) = -\frac{1}{2} \int \langle k_x k^{-1}, v \rangle dx .$$

We can calculate the Poisson brackets between these nice functions quite easily:

$$\{f_v, f_w\} = f'_v(k, wk) = f_{[v,w]} - \frac{1}{2} \int \langle w_x, v \rangle dx .$$

We now see the appearance of a central extension term in the Lie algebra. The Poisson brackets can be written as $\{f_v, f_w\} = f_{[v,w]} + \vartheta(v, w)f_c$, where $f_c(k) = 1$ and the cocycle $\vartheta(v, w) = -\int \langle w_x, v \rangle dx / 2$. We can also manufacture a derivation term, which corresponds to the momentum (the operation of incrementing the x coordinate). Consider

$$\begin{aligned} \omega(k; k_y, k_x) &= -\frac{1}{2} \int \langle (k^{-1}k_y)_x, k^{-1}k_x \rangle dx = -\frac{1}{2} \int \langle (k_x k^{-1})_y, k_x k^{-1} \rangle dx \\ &= -\frac{1}{4} D_{(k;y)} \int \langle k_x k^{-1}, k_x k^{-1} \rangle dx . \end{aligned}$$

Thus the momentum is given by

$$f_d(k) = -\frac{1}{4} \int \langle k_x k^{-1}, k_x k^{-1} \rangle dx .$$

A brief calculation shows that $\{f_d, f_v\} = f_{v'}$ and $\{f_d, f_c\} = 0$.

7.2 Adjoint symmetries of the model and dual model

In this section we consider the left multiplication symmetry again, however this time we can simultaneously describe the action on the dual models. This requires some care with the boundary conditions, and we shall take the double Neumann condition on loops, i.e. $k = e$ and $k_x = 0$ at both boundaries. The operation of left multiplication by constants does not preserve these conditions, but we can use our freedom to introduce a right multiplication to work with the adjoint action instead.

Take the action on the phase space given by Ad_d for $d \in D$. This preserves the boundary conditions, and preserves the models in the case where $\text{Ad}_d \mathcal{E}_\pm = \mathcal{E}_\pm$. The corresponding infinitesimal motions are generated by the moment map

$$I_\delta(k) = -\frac{1}{2} \int \langle k_x k^{-1}, \delta \rangle dx, \quad \delta \in \mathfrak{d}.$$

If the map ad_δ preserves the subspaces \mathcal{E}_\pm then this formula gives conserved charges for the system.

7.3 Automorphism symmetries of the model and dual model

Here we consider symmetries of the phase space arising from group automorphisms $\theta : D = G \ltimes M \rightarrow D$. This is really a generalisation of the previous subsection, where we just considered automorphisms given by the adjoint action, i.e. inner automorphisms. We consider the same boundary conditions as in the last subsection. For convenience we also assume that the two subspaces $\theta \mathcal{E}_\pm$ of the Lie algebra \mathfrak{d} are perpendicular for the given inner product. This is not really needed, as we can always manufacture a new Ad-invariant inner product from the old one using the automorphism in order to make this true.

Given these conditions, any automorphism $\theta : D \rightarrow D$ will induce a map $\tilde{\theta}$ on the phase space given by $(\tilde{\theta}k)(x) = \theta(k(x))$. This map will be symplectic if θ preserves the given inner product on \mathfrak{d} , and if $\theta \mathcal{E}_\pm = \mathcal{E}_\pm$ then the map will preserve the given models. In general $\tilde{\theta}k$ will factor to give G -models and dual M -models which are a mixture of the original G -models and dual M -models given by factoring k . However there are two special cases worthy of mention.

1) The automorphism $\theta : D \rightarrow D$ is called subgroup preserving if $\theta G \subset G$ and $\theta M \subset M$. In this case a factorisation $k = us$ for $u \in G$ and $s \in M$ is sent to $\theta(k) = \theta(u)\theta(s)$, and $\theta(u)$ is

a solution of the sigma model on G . In the same manner, if t is a solution of the sigma model on M , then $\theta(t)$ is also a solution of the sigma model on M .

2) The automorphism $\theta : D \rightarrow D$ is called subgroup reversing if $\theta G \subset M$ and $\theta M \subset G$. If such an automorphism exists, the double $D = G \bowtie M$ is called self-dual[22]. In this case a factorisation $k = us$ for $u \in G$ and $s \in M$ is sent to $\theta(k) = \theta(u)\theta(s)$, and $\theta(u)$ is a solution of the sigma model on M . In the same manner, if t is a solution of the dual sigma model on M , then $\theta(t)$ is also a solution of the sigma model on G . In this manner the solutions of the sigma model on G and the dual sigma model on M are related by a group homomorphism from G to M , and in that sense the models are self-dual.

Other symmetries may be constructed. For example of we have $\theta\mathcal{E}_+ = \mathcal{E}_-$ and $\theta\mathcal{E}_- = \mathcal{E}_+$ then the map $\hat{\theta}(k)(t, x) = \theta(k(t, \pi - x))$ sends a solution k of the model into another solution.

The explicit computation of examples of our generalised T-duality along the above lines is a topic for further work. However, the data required for the construction do exist in abundance. For example, given any two Lie algebras $\mathfrak{g}_0 \subset \mathfrak{d}$ whose Dynkin diagrams differ by the deletion of some nodes, one has an inductive construction $\mathfrak{d} = (\mathfrak{n} \bowtie \mathfrak{g}_0) \bowtie \mathfrak{n}^*$ where \mathfrak{n} are braided-Lie bialgebras [21]. For a concrete example, one has, locally,

$$D = SO(1, n + 1) = (\mathbb{R}^n \bowtie SO(n)) \bowtie \mathbb{R}^n$$

as the decomposition of conformal transformations into Poincaré and special conformal translations. The group D has a non-degenerate bilinear form as required (although not positive-definite). The explicit construction of the required factorisation and the associated bicrossproduct quantum groups and T-dual models will be attempted elsewhere.

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